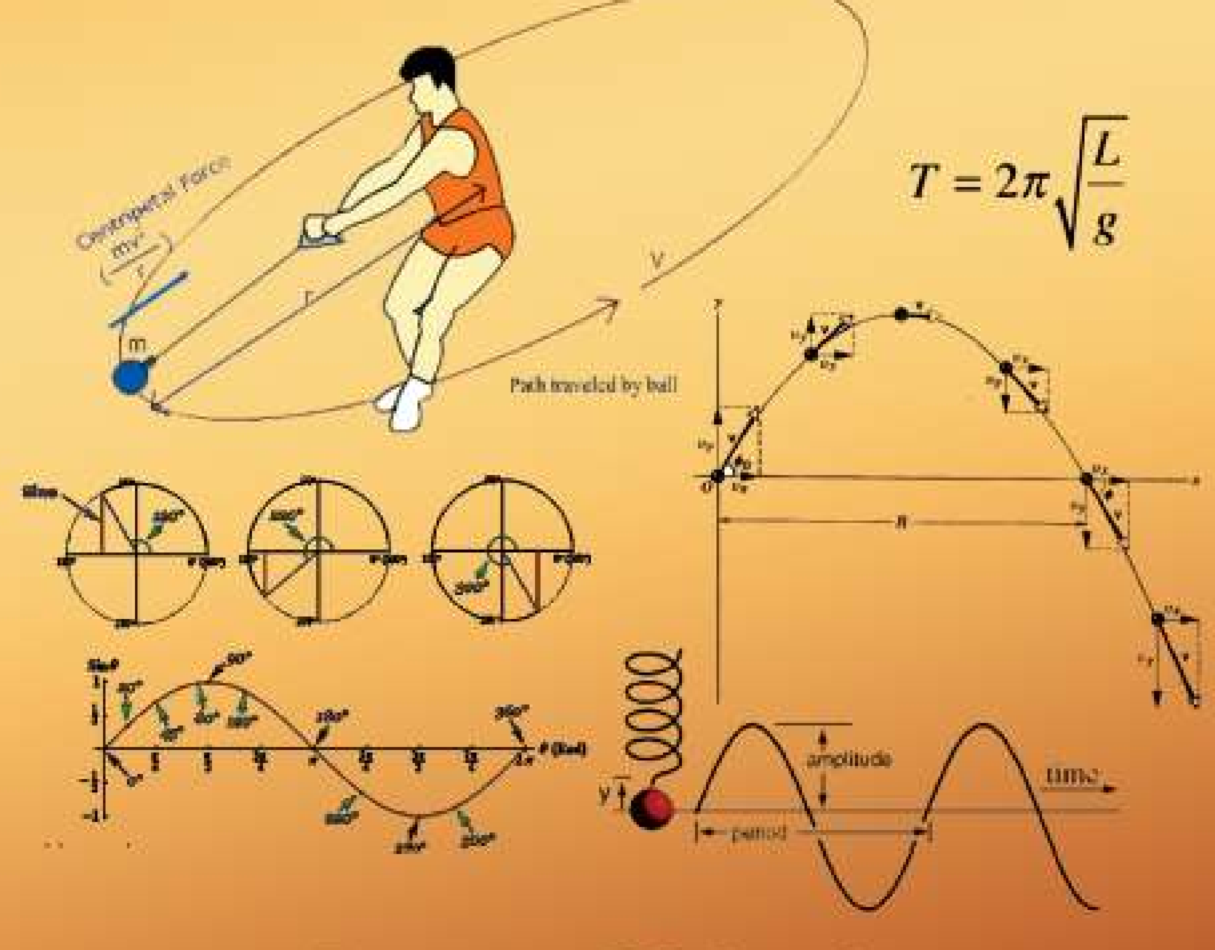
General Certificate of Education (Advanced Leval)

Combined Mathematices

Practice Questions with Answers

(Prepared to siut the new syllabus implemented from 2017)



Department of Mathematics
Faculty of Science and Technology
National Institute of Education
Sri Lanka

G.C.E. Advanced Level

Combined Mathematics

PRACTICE QUESTIONS

WITH ANSWERS

(Prepared to suit the syllabus implemented from 2017)

Department of Mathematics
National Institute of Education
Maharagama
Sri Lanka

Combined Mathematics
G.C.E. Advanced Level

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Message from the Director General

The Department of Mathematics of the National Institute of Education (NIE) has adopted a variety of methodologies with a view to promoting mathematics education. This book entitled "Practice questions with answers" is a result of such an exercise.

Since the General Certificate of Education (Advanced Level) examination is a highly competitive examination, preparing the students is also a highly competitive task for the teachers, who teach Combined Mathematics for grade 12 and 13. To overcome the difficulty of this task teachers need supportive materials which are developed according to a proper standardand. But such type of materials are quite rare. It is not a secret that most of the instruments available in the market are composed of questions that lack validity and quality. The Department of Mathematics of the NIE has prepared this Practice questions with answers to rectify this situation and facilitate students to prepare well for the examination. This collection comprises of questions prepared according to the syllabus implemented from 2017. Inclusion of answers along with questions undoubtedly, makes it easier to use for the teachers.

I request teachers and students to make the evaluation process in Combined Mathematics a success by having access to this book.

I wish to extend my gratitude to the donors of the Austraian Aid programe for assistance given to make this book available to you and also to the staff of the Department of Mathematics and external scholars who provided academic contribution to make this venture a success.

Dr. (Mrs.) T. A. R. J. Gunasekera
Director General
National Institute of Education

Preface

Among the G.C.E. (A/L) subject areas there is a special place belonging to the area of mathematics Most of the students who complete G.C.E.(O/L) with high grading wishes to continue their education in maths stream. The past evidence shows that most of the creative inventors came from the field of mathematics or related field of mathematics.

The syllabus prepared for G.C.E.(A/L) maths stream is with the intention of producing experts in the field of Mathematics, Science and Technology.

The revised new syllabus for Mathematics and Combined Mathematics was introduced from the year 2017.To make the learning of students easy, a book named 'practice question with answers' was prepared by "The Department of Mathematics of the National Institute of Education".

The way of questions provided in this book will help students to make them feel more comfortable and also help them to prepare for their G.C.E.(A/L) examination by self-measuring their achievement level. For the above reason It is expected to develop skills and ability on writing necessary steps while answering the question in the public examinations by gaining experience by practicing questions on this book.

I kindly request you to send feedback to us about the benefits you gained by using this practice questions at your schools. It will be useful to us to edit and publish future editions of this book.

I earnestly thank our Director General for granting permission to prepare such a book and also thank the resource people contributed in the making of this book. Again, I request you to use this book in a proper and prospective way and send your valuable positive criticism and comments about this book it will motivate us to produce more books.

Mr. K. R. Pathmasiri
Director
Department of Mathematics.

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Introduction

This book is prepared with the intend of giving more practice and revision in order to get ready for the G.C.E.(A/L) final examination from the year 2019 onwards. After completing learning of the syllabus, students can test their knowledge by practicing the questions in this book. The subject teachers and students should note that this is not a book consisting of model questions but rather, a collection of practice questions.

After practicing the given questions, students can check and compare their solutions with the solutions provided, although the solutions obtained by the students need not to be exactly the same as the solutions given in the book .The solutions provided can be considered as guidelines for the student to learn how to obtain the answer. Also note that the solutions given here are to check and follow steps needed to present the proper solution in proper ways.

Although this “Practice Questions with Answers” book is prepared to facilitate the students siting for the G.C.E.(A/L) examination from 2019 onwards, under the revised syllabus implemented from 2017, students who study Mathematics or Higher mathematics can also use relevant parts of this book.

We have planned to publish the “Statics – I”, “Statics – II” and “Unit wise Practice Questions Book I and II”, followed by the publication of this book, “Practice Questions with answers”. Please, do not hesitate to point out the short comings and weaknesses of this book. Your comments will help us to improve the quality of the book. Moreover, we want to stress out that we will highly appreciate your valuable comments.

Thank You.

Mr. S. Rajendram
Project Leader
Grade 12-13 Mathematics.

Content

Combined Mathematics I

Part A

1. Solve : $2\left(x^2+\frac{1}{x^2}\right)-\left(x-\frac{1}{x}\right)-14=0$

2. Solve : $\sqrt{3x+1}-\sqrt{2-x}=\sqrt{2x-1}$

3. Show that $\log_9\left(xy^2\right)=\frac{1}{2}\log_3 x+\log_3 y$

   $\left(\text{Hint : } \log_b a=\frac{\log_c a}{\log_c b}\right)$

   Hence Solve the Simultaneous equations

   $\log_9(xy^2)=\frac{1}{2}$

   $\log_3 x.\log_3 y=-3$

4. Let $f(x)=3x^3+Ax^2-4x+B$; where *A, B* are constants. Given that $(3x+2)$, is a factor of $f(x)$ and when $f(x)$ is divided by $(x+1)$ the reminder is 2
   (i) Find the values of *A* and *B*
   (ii) Express $f(x)$ as a product of lincer factors

5. Let $f(x)=x^4+hx^3+gx^2-16x-12$; where *h* and *g* are constants. Given that $(x+1)$ is a factor of $f(x)$ and when $f(x)$ is divided by $(x-1)$ the reminder is -24
   (i) Find the values of *h* and *g*
   (ii) Show that $(x-2)$ factor of $f(x)$ and find the remaining liner factors.

6. The roots of the equation $ax^2+bx+c=0$ are α and β . find the roots of the equation

   $x+2+\frac{1}{x}=\frac{b^2}{ac}$ in terms of α and β

7. If the equations $x^2+bx+ca=0$, and $x^2+cx+ab=0$ have a common root and *a, b, c* are all different, prove that their other roots will satisfy the equation $x^2+cx+bc=0$

8. If $g(x) = ax^2 - 2x + (3a+2)$, find the set of values of a for which $g(x)$ is positive for all real values of x

Sketch the graph of $y = g(x)$ when $a = \dfrac{1}{3}$

9. Find the solution set of inequality $\dfrac{12}{x-3} \le x+1$

10. Solve $|1-2x| - |x+2| \le 2$

11. How many ways can four boys and four girls sit in a row?
Find the number of ways, if
(i) Two particular girls do not sit together
(ii) No girls sit next to each other.

12. For what values of k does the coefficient of x^2 in the expansion of $\left(x^2 - \dfrac{2k}{x}\right)^{10}$ equal the coefficient of $\dfrac{1}{x}$

13. Find the coefficient of x^2 and x^3 in terms of k and n, in the expansion of $(1+2x+kx^2)^n$ where n is a positive integer. If the coefficient of x^2 and x^3 are 30 and 0 respectively. Find the values of k and n.

14. Let $Z = -1 + i\sqrt{3}$ be a complex number
(i) Find $|Z|$ and $Arg(Z)$
(ii) Express Z^2 in the form of $a+ib$ where $a, b \in \mathbb{R}$
(iii) Find the values of the real number p such that $Z^2 + pz$ is real.
(iv) Find the value of a real number q such that $Arg(z^2 + qz) = \dfrac{5\pi}{6}$

15. Let $Z_1 = 1, \quad Z_2 = \cos\theta + i\sin\theta (0 < \theta < \pi)$ be two complex numbers. Represent the complex numbers Z_1 and Z_2 by the points A and B respectively in the Argand diagram. Find the points C and D to represent the complex numbers $Z_1 + Z_2$ and $Z_2 - Z_1$ respectively. Using your diagram, find
(i) $|Z_1 + Z_2|$ and $Arg(Z_1 + Z_2)$ (ii) $|Z_2 - Z_1|$ and $Arg(Z_2 - Z_1)$

Deduct that $|Z_1 + Z_2|^2 + |Z_2 - Z_1|^2$ is independent of θ

16. (a) Find $\lim_{x \to a} \dfrac{\sin x - \sin a}{x - a}$

(b) If $\sin y = x \sin(y + a)$

Show that $\dfrac{dy}{dx} = \dfrac{\sin^2(y+a)}{\sin a}$ எனக் காட்டுக.

17. (a) Find $\lim_{x \to 0} \dfrac{\tan x - \sin x}{x^3}$

(b) If $y = x^n lnx$, find the value of n

Such that $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} = 3x^2$ for all values of x

18. If $x = t + \ln t$ and $y = t - \ln t$ $(t > 0)$

Find

(i) $\dfrac{dy}{dx}$ (ii) $\dfrac{d^2y}{dx^2}$ interms of t

Also show that $\dfrac{d^2y}{dx^2} = \dfrac{8(x+y)}{(x+y+2)^3}$

19. Simplify $\dfrac{1}{1+x^2} - \dfrac{1}{(1+x)^2}$

Hence find $\displaystyle\int_0^1 \frac{x}{(1+x^2)(1+x)^2}$

20. Use the substitution $x = 2(1 + \cos^2\theta)$

to evaluate $\displaystyle\int_2^3 \sqrt{\frac{x-2}{4-x}}\,dx$

21. Using the method of integration by parts,

find $\int e^{4x}.\cos 3x.dx$

22. The line $3x+2y=24$ meets the y axis at A and x axis at B. The perpendicular bisector of AB meets the line parallel to x axis through (0, -1) at C. Find the area of the triangle ABC.

23. The equation of a side of a square is $x-2y=0$ and its diagonal intersect at $\left(\frac{5}{2},\frac{5}{2}\right)$ find the equations of the remaining sides of the square.

24. ABC is a triangle and AB=AC. $A\equiv(0,8)$. The equations of the medians through B and C are $x+3y=14$ and $3x-y=2$ respectively. Find the equations of the sides of the triangle ABC

25. A straight line $x\cos\alpha+y\sin\alpha-p=0$ intersect in circle $x^2+y^2-a^2=0$ at A and B. Find the equation of the circle AB as a diameter.

26. S is the circle $S\equiv x^2+y^2-4x-2y+4=0$ and P is the point $P\equiv(4,2)$
   (i) Show that the point P lies outside S.
   (ii) Find the length of the tangents from P to S
   (iii) Find the equations of the tangents from P to S

27. Find the general equation of all circles which make an intercept 3 units on the x axis and touch the y axis.
Show that their centers lie on the curve whose equation is $4x^2-4y^2=9$

28. Solve : $\cos 6\theta+\cos 4\theta+\cos 2\theta+1=0$, where $0<\theta<\pi$

29. Show that $2\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{1}{7}\right)=\frac{\pi}{4}$

30. In a triangle ABC, with the usual notation,
Prove that $(b+c-a)(\cot\frac{B}{2}+\cot\frac{C}{2})=2a\cot\frac{A}{2}$

31. State De Movier's theorem by using De Movier's theorem find the modulas and argugment of $(1+\sqrt{3}i)^7$

32. Using De Movier's theorem prove that

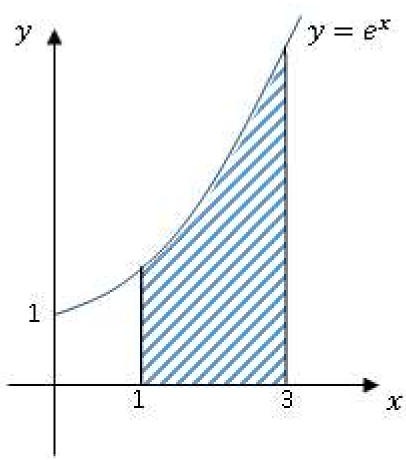
(i) $\cos 3\theta = 4\cos^3\theta - \cos\theta$

(ii) $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

33. Parametric equation of a curve is given by $x = t(1-t)^2,\ y = t^2(1-t)$ where t is a real parameter. Show that gradient of thetangent at the point $\left[t(1-t)^2,\ t^2(1-t)\right]$ is given by $\dfrac{t(2-3t)}{(1-t)(1-3t)}$ where $t \neq 1,\ \dfrac{1}{3}$ also show that equation of tangent drawn to the curve at the point corresponding to $t = \dfrac{1}{2}$ is $4x + 4y - 1 = 0$.

34. Find the area bounded by the curve $y = x(x-3)$ and the x axis.

35.



(i) Find the area shaded by the region.

(ii) Find the volume of the solid form by rotating the shaded region through four rectangles about x axis.

Part B

1. (*a*) The root of the equation $x^2 + px + q = 0$ are α and β.

   (i) Given that the roots differ by $2\sqrt{3}$ and the sum of the reciprocal of the roots is 4, find the possible values of p and q.

   (ii) find an equation whose roots are $\alpha + \dfrac{2}{\beta}$ and $\beta + \dfrac{2}{\alpha}$ expressing the coefficients in terms of p and q.

   (*b*) Find the possible values of k if $\dfrac{x^2+3x-4}{5x-k}$ can take all values when x is real.

   Draw the graph of $y = \dfrac{x^2+3x-4}{5x-k}$, when $k = -5$

2. (*a*) The roots of the quadratic equation $f(x) = \lambda^2(x^2 - x) + 2\lambda x + 3 = 0$, $(\lambda \neq 0)$ are α and β. If λ_1, λ_2 are the values of λ. For which α and β are connected by the relation $\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha} = \dfrac{4}{3}$, find the equation. Whose roots are $\dfrac{\lambda_1^2}{\lambda_2}$ and $\dfrac{\lambda_2^2}{\lambda_1}$.
   Find the greatest integer λ such that the quadratic function $f(x) > 2\lambda x$ for all values of x.

   (*b*) Prove by mathematical induction that $\displaystyle\sum_{r=1}^{2n} (-1)^{r+1}\frac{1}{r} = \sum_{r=n+1}^{2n} \frac{1}{r}$

3. (*a*) Express $\dfrac{2r+3}{r(r+1)}$ in partial fractions.

   Write the r[th] terms U_r of the series $\dfrac{5}{1.2}\left(\dfrac{1}{3}\right) + \dfrac{7}{2.3}\left(\dfrac{1}{3}\right)^2 + \dfrac{9}{3.4}\left(\dfrac{1}{3}\right)^3 + \ldots$

   Find V_r such that $U_r = V_r - V_{r+1}$

   Hence find $\displaystyle\sum_{n=1}^{\infty} U_n$. Is the series $\displaystyle\sum_{n=1}^{\alpha} U_r$ convergent. Justify your answer.

   (b) Sketch the graphs of $y = |2x-1|$ and $y = |x+1| + 1$ in same diagram.

   Hence solve $|2x-1| - |x+1| \geq 1$.

4. (a) Six boys and six girls sit in a row at random. Find the number of different ways that,

(i) The six girls sit together

(ii) The boys and girls sit alternatively.

(b) Four digit numbers are formed by choosing digits from 0, 2, 3, 5, 7, 8

How many numbers can be formed If

(i) Digits can be repeated in a number

(ii) One digit can be used once only in a number

incase (ii) how many numbers are greater than 5000 and divisible by 2.

(c) State and prove the binominal theorem for positive integral index.

Write down the binominal expansion of $(1+x)^n$ and $(x+1)^n$; where n is a positive integer by considering the first derivatives of both expansion, show that

(i) $$1(n-1){}^nC_1^{\,2} + 2(n-2){}^nC_2^{\,2} + \ldots + r.(n-r).{}^nC_r^{\,2} + \ldots + (n-1).1.{}^nC_{n-1}^{\,2}$$
$$= n^2.{}^{2n-2}C_{n-2}$$

(ii) $$\sum_{r=1}^{n} r \cdot {}^nC_r \cdot \sum_{r=0}^{n-1} (n-1) \cdot {}^nC_r = n^2 \cdot 2^{2n-2}$$

5. (a) Find the three roots of $Z^3 = 1$

Given that is one of the complex roots of $Z^3 = 1$, show that $1 + \omega + \omega^2 = 0$

Hence show that

(i) $$\frac{\omega}{\omega+1} = -\frac{1}{\omega}$$

(ii) $$\frac{\omega^2}{\omega^2+1} = -\omega$$

(iii) $$\left(\frac{\omega}{\omega+1}\right)^{3k} + \left(\frac{\omega^2}{\omega^2+1}\right)^{3k} = -2, \quad k \text{ is odd}$$
$$= +2, \quad k \text{ is even}$$

(b) Get $u = 2i$ and $v = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ be two complex numbers. Write u, v, uv, $\frac{u}{v}$ in the form $r(\cos\theta + i\sin\theta)$ where $(-\pi < \theta \le \pi)$

In an argand diagram the points *A, B* and *C* represent the complex numbers u, uv and $\frac{u}{v}$ respectively. Show that *ABC* is an equilateral triangle.

6. (a) Express $\left(\dfrac{1+i}{1-i}\right)^{4n+1}$ in the form of $p+iq$ where $p,q\in\mathbb{R}$; and n is a positive integer.

Show that the cube root of 1 is $1,\omega,\omega^2$

Where $\omega=\cos\dfrac{2\pi}{3}+i\sin\dfrac{2\pi}{3}$

Hence solve the equation $(x+2)^3=1$

Also show that

(i) $\left(2+5\omega+2\omega^2\right)^6=729$

(ii) $(p-q)(p\omega-q)\left(p\omega^2-q\right)=p^3-q^3$

(iii) $\left(\dfrac{a+b\omega+c\omega^2}{b+c\omega+a\omega^2}\right)=\omega$

(b) The point $P(x,y)$ denotes the complex number $Z=x+iy$ Argand diagram, where $x,y\in\mathbb{R}$

Given that $|Z-3-3i|=2$ find the locus of P and sketch it in the argand diagram

Further if $0\le Arg(Z-3-3i)\le\dfrac{\pi}{3}$ shade the region which satisfies both conditions in the Argand diagram. Also find the greatest value of $|Z|$ in their region.

7. (a) Find

(i) $\lim\limits_{x\to 0}\dfrac{\cos 4x-\cos^2 x}{x^2}$

(ii) $\lim\limits_{x\to 0}\dfrac{\tan 2x-2\sin x}{x^3}$

(b) Given that $y=\sin^{-1}\dfrac{1}{\sqrt{x^2-1}}$, $Z=\sec^{-1}x\left(x>\sqrt{2}\right)$

Show that

(i) $\cos y.\dfrac{dy}{dz}=-\cos ec^2 z$

(ii) $\dfrac{dy}{dz}+\dfrac{x^2}{\sqrt{\left(x^2-1\right)\left(x^2-2\right)}}=0$

(c) A wire of length l is bent in the shape of isosceles triangle. Show that the maximum area included in the triangle is equilateral and find the maximum area.

8. (*a*) If $f(x) = \sin 2x$ prove from first principle that $\frac{d}{dx}[f(x)] = 2\cos 2x$

Using the principle of mathematical induction prove that

$$\frac{d^n}{dx^n}(\sin 2x) = 2^n \sin\left[\frac{n\pi}{2} - 2x\right]$$

(*b*) Let $f(x) = 1 + \frac{1}{x^2 - 2x}$ Where $x \neq 0, 2$

Find the turning points of the graph of $f(x)$ only by **using first derivatives** sketch the graph of $y = f(x)$ indicating the asymptotes and maxima or minima (if any) Hence, sketch the graph of

(i) $y = |f(x)|$

(ii) $y = \frac{1}{f(x)}$

9. (*a*) Express $\frac{1}{(1-x^2)(x^2+1)}$ in partial fractions.

Hence, find $\int \frac{dx}{(1-x^2)(x^2+1)}$

(*b*) Given that $\sin x - \cos x = t$ express $\sin 2x$ in terms of *t*.

Using the above substitution, evaluate $\int_0^{\frac{\pi}{4}} \frac{\sin x + \cos x}{9 + 16\sin 2x} dx$

(*c*) $I = \int_0^{\frac{\pi}{2}} \frac{\cos x}{a\cos x + b\sin x} dx, \quad J = \int_0^{\frac{\pi}{2}} \frac{\sin x dx}{a\cos x + b\sin x}$

(i) Find $aI + bJ$

(ii) By obtaining another linear combination in I and J, hence find the values of I and J.

10. (*a*) Prove that $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

Show that $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx = \frac{\pi^2}{4}$

(*b*) Using the method of integration by parts,

Find $\int \frac{x.e^x}{(1+x)^2}dx$

(*c*) Find the area bounded by the curve $y = x(2-x)$ and the straight line $y = x$

11. (*a*) A rectangle *ABCD* lies completely in the first quadrant. The equation of *A*D is $x + y - 4 = 0$ and the equation of *AC* is $3x - y - 8 = 0$ and length of AB is $2\sqrt{2}$

(i) Find the equation of AB

(ii) Find the coordinates of B

(iii) If BD is parallel to $x - 3y + 7 = 0$ find the circle equations of BC and CD.

(b) Show that the general equation of the circle $S = 0$ passing through the points (2, 0) and (0, -1) is

$$S \equiv x^2 + y^2 - \left(\frac{\lambda+4)}{2}\right)x + (\lambda+1)y + \lambda = 0$$

Where λ is a parameter.

(i) Hence find the equation of the circle $S_1 = 0$ which passes through the points (1, -1), (2, 0) and (0, -1)

(ii) $S_1 = 0$ bisects the circumference of the circle $S_2 = 0$ of the given system $S = 0$ find the equation of $S_2 = 0$

(iii) Two circles of the above system $S = 0$ bisects orthogonally each other. Show that $\lambda_1\lambda_2 = -4$ where, λ_1 and λ_2 are the corresponding parameters of the circles.

12. (*a*) In a triangle *ABC*, the equation of internal bisector of *C* is $x-4y+10=0$ and the equation of the median through *B* is $6x+10y-59=0$

The coordinates of A is (3, -1) find

(i) the coordinates of *B* and *C*

(ii) the equations of the sides of the triangle *ABC*

(iii) the equation of the perpendicular to *AC* through *B*

(b) A circle $S_3=0$ passes through the points of intersection of the circles $S_1 \equiv 3x^2+3y^2-6x-1=0$, $S_2 \equiv x^2+y^2+2x-4y+1=0$ and also passes through the centre of $S_1=0$

Find the equation of $S_3=0$ and verifies that $S_3=0$ and $S_2=0$ intersect each other orthogonally.

Find also the equation of the tangent to the circle $S_3=0$ at the centre of $S_1=0$

13. (*a*) Find the general solutions of the equations.

(i) $(2\sin x-\cos x)(1+\cos x)=\sin^2 x$

(ii) $2\tan x+\sec 2x=2\tan 2x$

(b) Prove that $2\cos^2\theta-2\cos^2 2\theta=\cos 2\theta-\cos 4\theta$ and deduct that

$\cos 36^0-\cos 72^0=\dfrac{1}{2}$

Hence, find the values of $\cos 36^0$ and $\cos 72^0$

(c) State and prove the since rule for a triangle ABC, with the usual notation.
In the usual notation for a triangle ABC,

(i) show that $\dfrac{a^2-b^2}{\cos A+\cos B}+\dfrac{b^2-c^2}{\cos B+\cos C}+\dfrac{c^2-a^2}{\cos C+\cos A}=0$

(ii) if $A=45^0$ and $B=75^0$, show that $a+\sqrt{2}c=2b$

14. (*a*) Solve the equation

(i) $2(\cos x+\cos 2x)+\sin 2x(1+\cos x)=2\sin x$
where $-\pi<x\le\pi$

(ii) $\tan^{-1}\left(\dfrac{1}{x-1}\right)-\tan^{-1}\left(\dfrac{1}{x+1}\right)=\tan^{-1}\left(\dfrac{3}{5}\right)-\tan^{-1}\left(\dfrac{1}{3}\right)$

where $(2<x<4)$

(*b*) If $(1+m)\sin(\theta+\alpha)=(1-m)\cos(\theta-\alpha)$, prove that

$$\tan\left(\frac{\pi}{4}-\theta\right)=m\cot\left(\frac{\pi}{4}-\alpha\right)$$

(*c*) state and prove the cosine rule for a triangle *ABC*, with the usual notation.

(i) In a triangle ABC, AH is perpendicular to BC and $AH=p$ show that

$$(b+c)^2=a^2+2ap\cot\frac{A}{2}$$

(ii) If $a^4+b^4+c^4=2c^2(a^2+b^2)$ Prove that $C=45^0$ or 135^0

15. (*a*) Let $A=\begin{pmatrix}3 & 1\\ -1 & 2\end{pmatrix}$ be a 2×2 matrix.

Show that $A^2-5A+7I=0$; I is the identity matrix of order 2.
Hence Find A^{-1}
Also furthe matrix B of order 2 such that BA = C

Where $C=\begin{pmatrix}9 & -4\\ 6 & 16\end{pmatrix}$ Find B

(*b*) *x, y* are connected by equations $x-y=a$, $x+y=b$. Write down the equation in the form AX = B, where A, X, B are matrices.

Find A^{-1}
Hence find *x, y* interms of a, b.

It is given that $A^2\begin{pmatrix}p\\ q\end{pmatrix}=B$, without finding $(A^2)^{-1}$ only by using matrices find p, q interms of a and b.

Combined Mathematics II

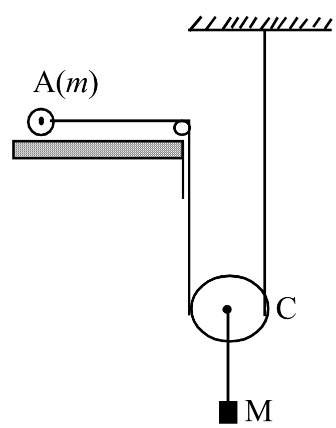
Part A

1. A train runs between two stations A and B which are 10km a part. It starts from A with an initial velocity u and uniform accetertaion $1ms^{-2}$. It moves for 40 seconds and reaches the speed 60 ms^{-1} and maintains the speed for T Seconds. It comes to rest at B with uniform retardation $\frac{1}{2}ms^{-2}$.
   (i) Draw the velocity - time graph for the motion of the train.
   (ii) From the graph find u and T

2. A particle A is projected vertically upwards with velocity u when A reaches its highest point, another particle B is projected vertically upwards with velocity $2u$ from the same point.
   (i) Draw the velocity - time graph for A and B in the same diagram.
   (ii) Find the time taken for the particles to meet after B is projected.

3. A ship A is travelling due east at $2u\ kmh^{-1}$ and a second ship B is travelling S 30^0 E at $u\ kmh^{-1}$. At midday the first ship is $d\ km$ due south of the second
   Find
   (i) The velocity of A relative to B.
   (ii) The least diatance between the two ships and the time taken.

4. A particle a of mass m rest on a smooth horizonal table and is connected by a light unextensibel string passing over a smooth fixed pulley at the edge of the table and under a smooth light pulley C to a fixed point on the ceiling as shown in the diagram. The pulley C carries a particle of mass M. If the system is released from rest. find the acceleraton of C and the tension in the string.



5. At time t the position vector of a particle is $\underline{r}$, $\underline{r} = a\cos nt\underline{i} + b\sin nt\underline{j}$ where $a, b, (a \neq b)$ and n are constants and $\underline{i}$ and $\underline{j}$ are unit vectors along the Ox, Oy axes respectively. Find $\underline{v}$, the velocity vector and $\underline{a}$ the acceleration vector and hence find the times at which the velocity is perpendicular to acceleration.
   Also show that $\underline{v}.\underline{v}. = n^2\left(a^2 + b^2 - \underline{r}.\underline{r}\right)$

6. A car *of mass 1200 kg* moves along a strainght horizontal road with a constant speed of *24 kmh*$^{-1}$ The resistance of motion to the car has magnitude *600N.*
   (i) Find, in *kW* the rate at which the engine of the car.
   (ii) The car now moves up a hill inclined at α to the horizontal, here $\sin\alpha = \frac{1}{24}$ The resistance to movion from non - gravitationnal forces remains of magutude *600N* The engine of the car now works at the rate of $30\ \text{kW}$. Fine the accelaraton of the car when its speed is *20 ms*$^{-1}$

7. Show that the velocity of water in a pipe of cross section *100 cm*^{2} which delivers *0.1m*^{3} *is 10 ms*$^{-1}$.
   Calculate the power of an engine which raise the water in this pipe to a height of *12m and then delivers at this hight at 10 ms*$^{-1}$ *(neglect friction)*

8. A gun of mass *M* is mounted on a smooth railway, and is fired in the direction of the track. It fires a shell of mass *m,* with velocity *v* relative to the gun. If the angle of elevation of the gun is α , prove that the unitial direction of the motion of the shell is $\tan^{-1}\left[\frac{M+m}{M}\tan\alpha\right]$ to the horizontal.

9. Three particles *A, B, C* of masses *m, 2m, 3m* respectively lie at rest in that order in a straight line on a horizontal table. The distance between consecutive particles is a. A slack light in elastic string of lenght $2a$ connects *A* and *B*. An exaclty similar string connects *B* and *C*. If *A* is prjected in the direction *CBA* with speed *v* find the speed with which C begins to move after the two springs become tant that the ratio of the unipulsire tensions in *BC* and *AB* when *C* is jerked into motion is 3:1. Find also the total loss of kinetic energy when *C* has started to move

10. Two small uniform smooth spheres *A* and *B* of equal size and of masses *m, 4m* and *4m* respectively are moving directly towardseach other with speeds *2u* and *6u* respectively. The coefficient of restitution between the spheres is $\frac{1}{2}$.
    Fine,
    (i) The speed of *B* Unimediatley after collion.
    (ii) The momentum transfered from one to other.

11. Two partcles A and B move on a smooth horizontal table. The mass of A is m, and the mass of B is $4m$. Initially A is moving with speed u when it collides directly with B, which is at rest on the table. As a result of the collision, the direction of motion of A is reversed. The coefficient of restitution between the particles is e. Find expressions for the speed of A and the speed of B unimediately after colision. In the susequent motion, B strikes a smooth vertical wall and rebound. The wall is perpendicular to the direction of motion of B. The coefficient of restitution between B and the wall is $\frac{4}{5}$. Given that there is a second collision between A and B, show that $\frac{1}{4} < e < \frac{9}{16}$.

12. A vertical cliff is $73.5m$ high. Two stones A and B are projected simultaneously. Stone A is projected horizontally from the top of the cliff with speed $28ms^{-1.}$ Stone B is projected fromthe botton of the cliff with speed $35ms^{-1}$ at an angle α above the horizontal. The stones move freely under gravity in the same vertical plane and collide in mid - air.

(i) Prove that $\cos\alpha = \frac{4}{5}$.

(ii) Find the time which elapses between the mistant when the stones are projected and the mistant when they collide. ($g = 9.8ms^{-2}$)

13 .A projectile is fired with unitial speed $\sqrt{2ag}$ to hit a target at a horizontal distance a from the point of projection and at a vertical distance $\frac{a}{2}$ above at. Find the two possible angles of projection and the ratio of the time of flight along the two paths.

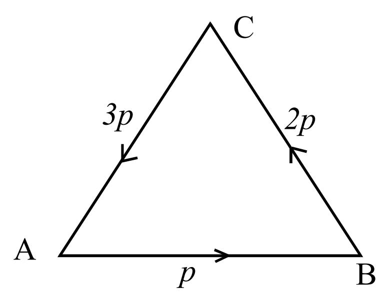
14. An elastic string AB of natural length a and modulus of elasticity $2mg$ has one end A fixed. A particle of mass m is attached to the end B and performs horizontal circles with angular velocity $\sqrt{\frac{3g}{4a}}$. Find the extension in the string and cosine of the angle between the string and the vertical.

15. A small bead of mass 2kg is threaded on to a smooth circular wire of radions $0.6m$, which is fixed in a vertical plane. If the bead is slightly disturbed from rest at the highest point of the wire, find its speed when it reaches the lowest point. Find also the height above the centre, of the print at which the reaction between the bead and the wire becomes zero. ($g = 10ms^{-2}$)

16. A particel is moving in a straight line with Simple Harmonic Motion. Its velocity has the values $1.2m$ and $0.9m$ when its distances from the centre of ocillation are $0.9ms^{-1}$ and $1.2ms^{-1}$ respectively.
Find the amplitude and period of the motion.

17. A particle of mass m is attached to the mid point of an elastic string of natural length a and modulus $2mg$. The ends of the string are fixed to two points in a vertical line at a distance of $2a$. positon of equilibriun both parts of the string are intension. If the partcle is given small vertical displacement and it performs simple hrmonic oscillation, find the period.

18. ABC is an equilaberal triangle of side $2a$. Forces $p, 2p$ and $3p$ act along $\overrightarrow{AB}, \overrightarrow{BC}$ and $\overrightarrow{CA}$ respectively A.
Fine
(i) The magnitude and resultant of the system of forces.
(ii) The distance from A of the point where its line of action ents BA produced.



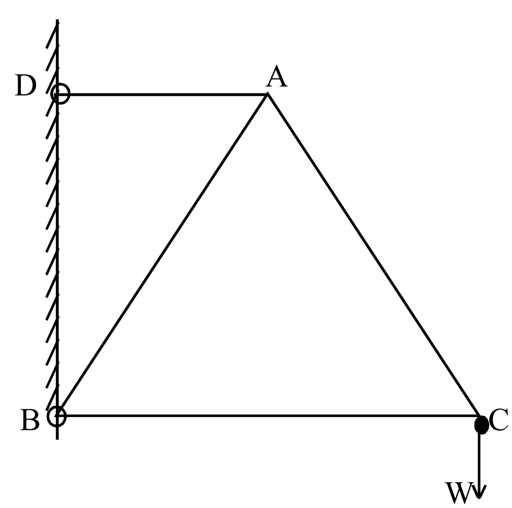
19. In the rectangle ABCD, $AB = 4a, and BC = 3a$. Forces $2p, 4p, 6p, 7p$ and $5p$ act along $\overrightarrow{AB}, \overrightarrow{BC}, \overrightarrow{CD}, \overrightarrow{DA}$ and $\overrightarrow{AC}$ ගrespectively. Show that the system reduces to a couple. Find the magnitude and sense of the couple. If the force acting along $\overrightarrow{BC}$ is removed find the magnitude, direction and the line of action of the resultant of the new system.

20. A uniform rod AB of length $2a$ and weight W is smothly pivoted at A to a fixed point. It is held in equilibrium at an angle $\tan^{-1}\left(\frac{3}{4}\right)$ to the downward vertical by a force of magnitude P applied at B.
(i) using triangle of forces Find P, if the force P is horizontal.
(ii) What is the least possible value of P and its direction.

21. A sphere of redus $9cm$ and weight W rests on a smooth inclined plane (angle 30^0). It is attached by a string fixed to a point on its surface to a point on the plane $12cm$ from the point of contact and on the same line of greatest slope. Mark the forces acting on the sphere.
Draw the triangle of forces for the equlibrium of the sphere and hence find.
(i) the tension in the string
(ii) Raction on the sphere by the plane

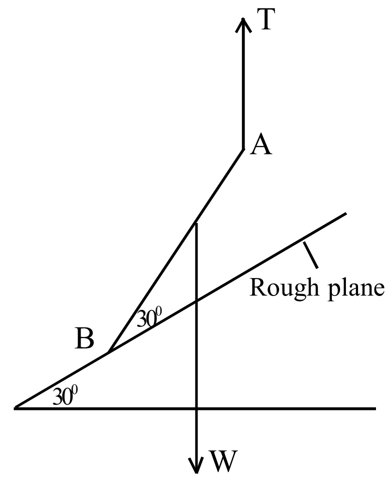
22. Three uniform rods AB, BC and CA of equal length a and weight W are feely jointed together to form a triangle ABC. The framewore rests in a vertical plane on smooth supports at A and C. so that AC is horizontal and B is above AC. A mass of weight W is attached to a point on D on AB where $AD = \frac{a}{3}$.
Find the reaction between the rods AB and BC.

23. A framewore consists of four light rods as shown in the diagram.
$AB = BC = CA = 2a, \ and \ AD = a$
It is smoothly hinged to a vertical wall at band D with BC horizontal, and carries a weight W at C. Using bow's notation draw a stress diagram and find the stress in each rod. Distaquish tension and thrust.



24. A force P acting parallel to and up a rough plane of unclinaton α is just safficient to prevent a body of mass m. From sliding down the plane. A force $3P$ acting parallel to and up the same plane causes the same mass to be on the point of moving up the plane. If μ is the coefficient of friction show that $2\mu = \tan\alpha$.

25. A uniform rod AB of weight W is in equilibrium in a vertical plane as shown in the diagram. A vertical string is attached to A

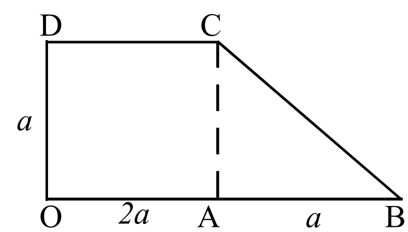


Fine

(i) T in terms of W

(ii) For equilibrium find the minimum value of μ, the coefficient of friction at B

26. A uniform lamina OABCD consists of a rectangle OACD and a right - angled triangle ABC as shown in the diagram. OA = 2a, OD = a, AB = a .



(i) Find the centre of gracityof the limina from OB and OD.

(ii) If it is feely suspended from O, find the angle OAB makes with the horizontal.

27. A and B are two events such that $P(B') = \frac{2}{3}$, $P(A \cup B) = \frac{5}{8}$ and $P(A | B) = \frac{3}{4}$.

Find $P(B)$, $P(A \cap B)$, $P(A)$ and $P(A' \cup B')$.

28. *(a)* Events A and B are such that $P(A) = 0.3$, $P(B) = 0.4$ and A and B are independent. Find

(i) $P(A \cup B)$

(ii) $P(A' \cap B')$

(b) If 20% of the bulbs produced by a machine is defective determine the probability that out of 4 balls chsen at raudom 3 will be defective.

29. The marks obtained by 9 students in an examination given below.

7, 11, 5, 8, 13, 12, 11, 9, 14

Fine

(i) Mean

(ii) Median

(iii) Standard deviation and

(iv) Coefficient of skewness

30. The age in years of the sesidents in a hotel is given below in the stem and leaf diagram.

0	2	(01)
1	1 5 7 9	(04)
2	1 3 8 9	(04)
3	2 3 3 5 6 6 7 9 9 9 9	(11)
4	0 5 7 7 8 9	(06)
5	8	(01)

2/3 Means 23 Years

(i) Write down the minimum value, maximum value end mode of the age of residen.

(ii) Find the values of Q_1, Q_3 and median.

(iii) Outlines are given by $Q_1 - 1.5(Q_3 - Q_1)$ and $Q_3 + 1.5(Q_3 - Q_1)$ check whether there is any outolines.

Part B

(01) (a) A particle P starts from rest, moves with uniform acceleration a in a straight line. After t seconds another particle Q. Starts from the same point with initial velocity u and uniform acceleration $\frac{3a}{2}$. Both particles move in the same direction and attam the same maximum velocity at the same time. Immediately they decelerate with uniform deceleration a and $2a$ respectively and come to rest.

Draw the velocity - timegraphs of P and Q in the same diagram.

Hence

(i) Show that the maximum speed is $3at - 2u$.

(ii) Show that the time difference in their hourney is $\frac{5t}{2} - \frac{u}{a}$

(iii) Find the distance travelled by each particle.

(b) Two straight roads OA and OB meet at an acute angle α. A car P moves along OA towards O with uniform speed u, while a second car Q moves along OB, away from O with uniform speed V. At $t = 0$, the car P is at a distance a from O and the car Q is at O. Find the relative velocity of P, relative to Q

(i) Show that the shortest distance between the cars is $\frac{av\sin\alpha}{\sqrt{u^2 + v^2 + 2uv\cos\alpha}}$ and find the time taken to reach shortest distance.

(ii) Show that the ratio of the distances from O when they are at shortest distance is $v + u\cos\alpha : u + v\cos\alpha$

(02) (a) A car weight W has maximum power H. In all circumstances there is a constant resistance R due to friction. When the cat is moving up a slope of $\sin^{-1}\left(\frac{1}{n}\right)$ its maximum speed is v and when it is moving down the same slope its maximum speed is $2v$.

Find R in terms of W and n.

The maximum speed of the car on the level road is u.

Find the maximum acceleration of the car when it is moving with speed $\frac{u}{2}$ up the given slop.

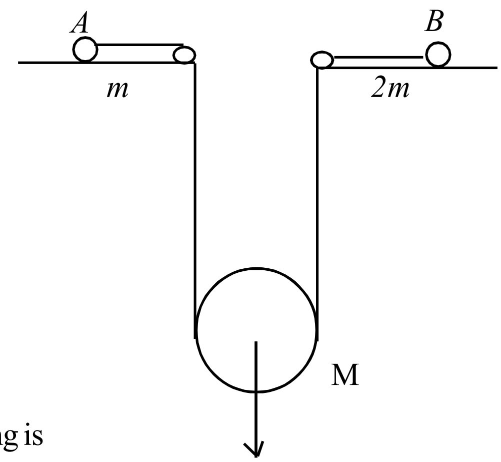
(b) Two particles A and B are free to move in the plane of the unit vectors $\underline{i}$ and $\underline{j}$ which are perpendicular to each other. The velocity of A is $\left(-3\underline{i}+2q\underline{j}\right)ms^{-1}$ and the velocity of B is $v\left(\underline{i}+7\underline{j}\right)ms^{-1}$ where v is a constant. Determine the velocity of B relative to A and fine the vector $\overrightarrow{AB}$ at time t seconds given that, when $t = O$, $\overrightarrow{AB} = \left(-56\underline{i}+8\underline{j}\right)m$

Find also the value of v such that the particles collide.

Show that, when $v = 3$, $\overrightarrow{AB}$ at time t is given by $\overrightarrow{AB} = \left(6t-56\right)\underline{i}+8(1-t)\underline{j}$ and hence find t where A and B are closest together.

By evaluating a suitable scalar product show that, for your value of t and with $v = 3$ $\overrightarrow{AB}$ is perpendicular to the velocity of B relative to A.

(03) (a) Two particles A and B of mass m and $2m$ are connected by a hight inextesible string passing under a smooth movable pulley of mass M. A and B rest on rough horizontal tables, as shown in the diagram, the coefficients of friction are μ and μ' respectively. The system is released from rest.



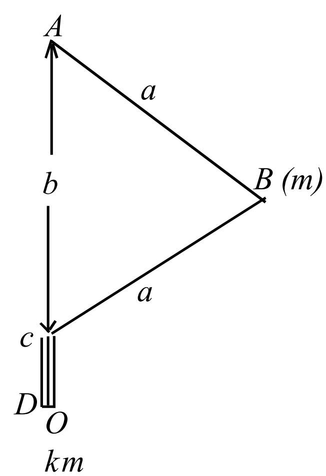
(i) Show that the tension in the string is

$$\frac{2Mmg\left(2+\mu+\mu'\right)}{\left(3M+8m\right)}$$

(ii) Given that $\mu > 2\mu'$, show that for the motion to take place.

$$\frac{\mu}{\mu'+2} < \frac{M+8m}{2M}$$

(b) One end of a light uniextensible string $ABCD$ in which $AB = BC = a$ is attached to a fixed point A. Asomooth narrow tube CD is fixed below A so that ACD is a vertical line and $AC = b$. The end D of the string is threeaded through the tube and attached to a body of mass km which cannot pass through the tube. A particle of mass m is fastened to the string at B, and rotates about the line AC with constanta angular velocity w in a horizontal circle,



tube at D. find the tensions in the two parts of the string and the vertical force exerted at D by the tube on the body, and show that $w^2ab \geq 2g(a+kb)$ Given that the greatest tension the string can sustain without breaking is λmg, show that the motion is possible only if $(\lambda - k)b \geq 2a$.

(04) (a) Three particle A, B, C of equal mass m lie at rest in that order in a straight on a smooth horizontal table. such that $AB = BC = d$. A is projected towards B with speed u and at the same time B is projectd towards C with the same speed u along the table. The coefficient restitution between any two particles is e,

(i) The time taken for A to collide with B

(ii) Find the distace travelled by A upto the above collision.

(iii) show that there is another collision between B and C.

(b) A particle P of mass m moves in a vertical circle along the smooth inner surface of a fixed. hollw sphere of internal radius a and centre O, the plane of the circle passing through O. The particle is projected from the lowest point of the sphere with a horizontal velocity u. Where $u^2 > 2ag$. When OP makes an angle θ with the upward vertical, the velocity of the particle is v and the normal reaction between the particle and the sphere is R. Find expressions for V *and* R in terms of m, a, u, θ *and* g show thati if $u^2 < 5ag$ the partcle leaves the shpere befor it reaches the hishest point of the sphere and find $\cos\theta$ in terms of u, a *and* g when it leaves the sphere. If the particle leaves the sphere at a point A and its trayctory meets the sphere again at a point B such that AB is a diameter of the shpere, show that OA makes an angle of 45^0 with the vertical, and find the requiste value of u.

(05) (a) A particle is projected at an angle α to the horizontal from a point at height h from horizontal ground. The particle reaches the ground at a point of horizontal distance $2h$ from the point of projection.

find the speed of projection in terms of g, α and h Given that the direction of motion of particle with horizontal when it reaches the group is β, show that t $\tan\beta = 1 + \tan\alpha$

(b) On a smooth inclime place of angle α there is placed a smooth wedge of mass M and angle α , in such a way that the upper face of the wedge is horizontal; on this horizontal face is placed *a* particle of mass *M.* The system is released from rest. Find the acceleraton of the wedge. Show that the reaction between the wedge and the plane is $\dfrac{M(M+m)g\cos\alpha}{M+m\sin^2\alpha}$

(06) Two points *A* and *B* on a smooth horizontal table are at a distance *8l* apart. A particle of mass *m* between *A* and *B* is attached to *A* by means of a light elastic string of modulus λ and natural length 2*l* and to B by means of a light elastic string of modulus 4λ and natural length 3*l.* If *M* is the midpoint of *AB, O* is the point between *M* and *B* at which the particle would rest in equilibrium, prove that $OM = \dfrac{2l}{11}$
If the particle is held at M and then released, show that it will move with simple harmonic motion, and find the period of motion.

Find the velocity of the particle when it is at a point *C* distant $\dfrac{3l}{11}$ from *M*, and is moving towards *B.*

(07) Particle of mass m is attached to one end of a light elastic string of natural length 6*a* and modulus of elasticity 3*mg*. The other end of the string is fixed to a point *O* on a smooth plane inclined at an angle 30^0 to the horizontal. The string lies along a line of greatest slope of the plane and the particle rests in equilibrium at a point *C* on the plane. calculate the distance *OC.*
The Particle is now pulled a further distance 2*a* down the line of greatest slope through *C* and released from rest. At time t later, the displacement of the particle from *C* is *x* is down the plane using the conservation of energy equation show that,

x Satisfies the differential equation. $\ddot{x} + \dfrac{gx}{2a} = 0$, untill the string becomes slack.

Given that $x = A\cos\omega t + B\sin\omega t$, where $\omega^2 = \dfrac{g}{2a}$, is the solution of above differential equation find *A* and *B* in terms of *a*.
Hence find the time at which the string slakckens and determine the speed of the partical at this time.

(08) (a) ABC is an equlaberal triangle of side $2a$. The moments of a system of foces acting in the plane of the triangle ABC, about A, B, and C are $M, \dfrac{M}{2}$ and $2M$ respectively in the same sense. Prove that the magnitude of the resultant of the system is $\sqrt{\dfrac{7}{12}}\dfrac{M}{a}$ and find tis direction with AB.

If the line of action of the resultant cut AB at D, find AD.

(b) A heavy uniform sphere of radius a has a light inextensible string attached to a point on its surface. The other end of the string is fixed to a point on a rough vertical wall. The sphere rests in equilibrium touching the wall at a point distant h below the fixed point. If the point of the sphere in contact with the wall is about to slip down wards and the coefficient of friction between the sphere and the wall is μ, find the inclination of the string to the vertica.

If $\mu = \dfrac{h}{2a}$ and the weight of the sphere is W, show that the tension in the string is $\dfrac{W}{2\mu}\sqrt{1+\mu^2}$

(09) (a) $ABCDEF$ is a regular hexagon with sides of length $2a$. Forces $P, P, Q, P\sqrt{3}\ N$ act along $\overrightarrow{AB}$, $\overrightarrow{DA}$, $\overrightarrow{CE}$ and $\overrightarrow{AE}$ respectively.

(i) Show that the system cannot reduce to a couple.

(ii) Find the resultant of the system when $Q = \sqrt{3}P$

(iii) If the line of action of the resultant cuts AB at G find AG

(b) Two equal uniform rods AB and BC, each of weight W are freely joined at B. The system is suspended frely from A and horizontal force P is applied at the lowest point C. If, in the equilibrium position, the inclination of AB to the down ward vertical is 30^0 find the corresponding inclination of BC and show that $P = \dfrac{W\sqrt{3}}{2}$

Determine the resultant action at B.

(10) (a) Two uniform beams AB and AC, equal in length and of weights $3W$ and W respectively, are smoothly jointed at A; the system rests in a vertical plane with the ends B and C in contact with a rough horizontal plane, the coefficient of limiting friction at B and C being the same and equal to μ.

If R and S are the normal reactions of the plane on AB *and* AC respectively and angle $B\hat{A}C = 2\theta$

(i) $R = \frac{5}{2}w, S = \frac{3}{2}w$

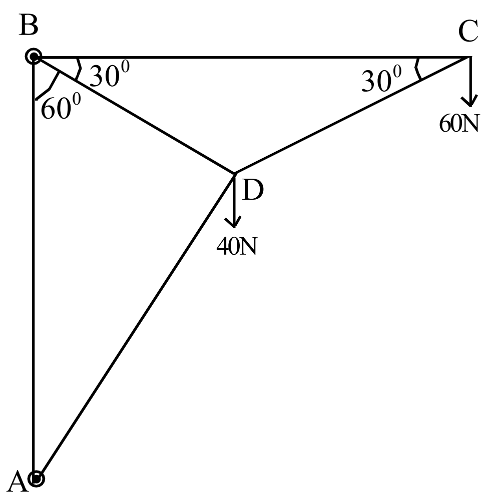
Stating at which point B or C the friction first becomes liniting as θ is in creased from zero.

(ii) Prove also that, $\tan\theta = \frac{3\mu}{2}$ the reaction of one beam on the other makes an angle of $\tan^{-1}(3\mu)$ with the vertical.

(b) In the framework shown, $BC = 6a$ The framework is hinged at A and is kept with BC horizontal by a force at B which acts downwards, perpendicular to *BD Weights 60N and 40N*

hang from C and D respectively find the magnitude and direction of the force a hinge A draw a stress diagram using bow's notation.

Hence, find the force in each rod distangnish - between thrust and tesions.



(i) Force at B.

(ii) Magnitude and direction of the force at the hinge.

(iii) By using Bow's notaion draw stress diagram and find forces in each rods also verify that tension and thrust.

(11) (a) $\underline{i}, \underline{j}$ are the unit vectors along Ox and Oy respectively forces $F_1 = 3\underline{i} + 4\underline{j}$, $F_2 = -\underline{i} + 6\underline{j}$, $F_3 = -3\underline{i} - 3\underline{j}$ act at the points whose position vectors are $r_1 = 2\underline{i} + 3\underline{j}$, $r_2 = 6\underline{i} + \underline{j}$, $r_3 = -3\underline{i} + 2\underline{j}$ respectively find the resultant force $\underline{R}$ and the cartesian equation of line of action of the resultant. If a fourth force $\underline{F_4}$, acting at the origin, and a couple $\underline{G}$ in the plane are added to the system to be in equlibrium find $\underline{F_4}$ and $\underline{G}$.

(b) In a triangle ABC, forces $\lambda\overrightarrow{BC}$, $\mu\overrightarrow{CA}$ and $\gamma\overrightarrow{AB}$ act along BC, CA and AB respectively. show that system of forces reduces to a couple if and only if $\lambda = \mu = \gamma$.

(c) The least force move a mass of $M\,kg$ up a plane of inclination α is P. Show that $P = Mg\sin(\lambda + \alpha)$ where λ is the angle of friction between the particle and the plane.
Show that the least force acting parallel to the plane which will move the mass up the slope is $P\sec\lambda$

(12) (a) A uniform circular lamina of radius a and weight W rests with its plane vertical on two fixed rough planes each incline at an angle α to the horizontal, their line of intersection being perpendicular to the plane of the lamina. If the coefficient of friction at each contact is μ, prove that the least couple required to rotate the lamina in the plane about its centre is of moment $\dfrac{\mu Wa}{\left(1+\mu^2\right)\cos\alpha}$

(b) A solid consists of a uniform right circular cone of density ρ, radius r; and height $4r$; mounted on a uniform hemisphere of density σ and radius r, , so that the plane faces cokncide. Show that the distance to the centre of mass of the whole solid from the common plance face is $\dfrac{r}{8}\left[\dfrac{16\rho - 3\sigma}{2\rho + \sigma}\right]$
If $\rho = \sigma$ and the solid is suspended frecly by a string attached to a point on the rim of the common plane face, find the inclination of the axis of the cone to the vertical.

(13) Show that the centre of mass of a uniform solid hemisphere of radus a is at a distance $\dfrac{3a}{8}$ from the centre.
A bowl is made by removing a hemisphere of radius a from a solid hemisphere of radius $2a$ both have the same centre O.
Find the distance of centre of mass of the bowl from O.

(i) The bowl is suspended from a point on the outer rim of the bow.. Show that the plane surface makes an angle α with the horizontal where

$$\alpha = \tan^{-1}\left(\frac{112}{45}\right)$$

(ii) If thes bowl rests in equilibrium with its curved surface in contact with a plane inclimed to the horizontal at an angle θ and sufficiently rough to prevent sliding, find the maxium value of θ.

(14) (*a*) Each sunday a fisherman visits on the three possible locations near his home; he goes to the sea with probability $\frac{1}{2}$; to a river with probability $\frac{1}{4}$; or to a lake with probability $\frac{1}{4}$. If he goes to the sea there is an 80% chnce that he will catch fish; corresponding fiqures for the river and the lake are 40%, 60% respectively.

(i) Find the probability that, on a given sunday, he catches fish.

(ii) Find the probability that he catches fish on at least two of three consecutive sundays.

(iii) If, on a particular sunday, he comes home without catching anything, where is it most likely that he has been.

(iv) His friend, who goes fishing every sunday, chooses among the three locations with equal probabilities. Find the probability that the two fishermen will meet at least once in the next two weekends.

(*b*) The table below gives the wages paid per hour and the number of employees of a factory.

Wages/ hour (In rupees)	**Number of employees**
900 - 800	14
800 - 700	44
700 - 600	96
600 - 500	175
500 - 400	381
400 - 300	527
300 - 200	615
200 - 100	660

Calculate

(i) the mean wage

(ii) Standard diviation

(iii) Median

(iv) Ciefficient of skewnees

and draw the shape of the distribution.

(15) (a) $\frac{3}{4}$ of a sports club are adults, and $\frac{1}{4}$ are chidre, Three quarters of the adults, and three fifth of the children, are male, Half the adult males, and on third of the adult females, use the swimming pool at the club; the corresponding proportion for children of either sex is four fifths.

(i) Find the probability that a memnber of a club uses the swimming pool.

(ii) Find the probability that a member of the club who uses the swimming pool is a female.

(iii) Find the probability that a male user of the swimming pool is child.

(iv) Find the probability that a member of the club who does not use the swimming poo is either female or an adult.

(b) A population consists of n_1 males and n_2 females. The mean height of the males and females are μ_1 and μ_2 respectively and the vaniance of the heights are $\sigma_1{}^2$ and $\sigma_2{}^2$ respectively.

Show that mean height of the whole population is $\mu_1 w_1 + \mu_2 w_2$ and the variance is $w_1\sigma_1^2 + w^2\sigma_2^2 + w_1 w_2 \left(\mu_1 - \mu_2\right)^2$

Where $w_1 = \dfrac{n_1}{n_1 + n_2}$ and $w_2 = \dfrac{n_2}{n_1 + n_2}$

The mean and standard deviation of a test for a group of 20 pupils were calculated as 40 and 5 respectively. But while calculating them a mark of 15 was misread as 50. Find the correct mean and standard deviation. The mean and standard deviation of another group of 30. pupils for the same lest is 40.25 and 8 respectively. Calculate the mean and standard deviation of the combined group of 50 pupils.

Solutions
for Practice Questions

Combined Mathematics I

Part A

1. $2\left(x^2+\frac{1}{x^2}\right)-\left(x-\frac{1}{x}\right)-14=0$

Let $y=x-\frac{1}{x}$

$x^2+\frac{1}{x^2}=y^2+2$

$2(y^2+2)-y-14=0$

$2y^2-y-10=0$

$(2y-5)(y+2)=0$

$y=\frac{5}{2}$ or $y=-2$

$x-\frac{1}{x}=-2$

$x^2+2x-1=0$

$x=\frac{-2\pm\sqrt{8}}{2}$

$x=-1\pm\sqrt{2}$

$x-\frac{1}{x}=\frac{5}{2}$

$2x^2-5x-2=0$

$x=\frac{5\pm\sqrt{41}}{4}$

2. $\sqrt{3x+1}-\sqrt{2-x}=\sqrt{2x-1}$

$x\geq-\frac{1}{3}$ and $x\leq 2$ and $x\geq\frac{1}{2}$

$\frac{1}{2}\leq x\leq 2$

Squaring both sides

$(3x+1)+(2-x)-2\sqrt{(3x+1)(2-x)}=2x-1$

$2=\sqrt{(3x+1)(2-x)}$

$4=(3x+1)(2-x)$

$3x^2-5x+2=0$

$(3x-2)(x-1)=0$

$x = \dfrac{2}{3}$ or 1

When $x = 1$, L.H.S $= \sqrt{4} - \sqrt{1} = 2 - 1 = 1$

R.H.S $= \sqrt{1} = 1$

L.H.S $=$ R.H.S

When $x = \dfrac{2}{3}$, R.H.S $= \sqrt{3} - \dfrac{2}{\sqrt{3}} = \dfrac{1}{\sqrt{3}}$

L.H.S $= \dfrac{1}{\sqrt{3}}$

R.H.S $=$ L.H.S

Hence, $x = \dfrac{2}{3}$ or 1

3. $\log_9\left(xy^2\right) = \log_9 x + \log_9 y^2$

$$= \frac{\log_3 x}{\log_3 9} + \frac{\log_3 y^2}{\log_3 9}$$

$$= \frac{\log_3 x}{2} + \frac{2\log_3 y}{2}$$

$$= \frac{1}{2}\log_3 x + \log_3 y$$

Let $\log_3 x = a$ and $\log_3 y = b$

$\log_9(xy^2) = \dfrac{1}{2}$ $\qquad$ $\dfrac{1}{2}a + b = \dfrac{1}{2}$

$a + 2b = 1$ ——— (1)

$\log_3 x.\log_3 y = -3$ $\qquad$ $ab = -3$ ——— (2)

From (1) and (2)

$b(1-2b) = -3$

$2b^2 - b - 3 = 0$

$(2b-3)(b+1) = 0$

If $b = \dfrac{3}{2}$, $a = -2$ $\qquad$ If $b = -1$, $a = 3$

$\left.\begin{array}{l} x = \dfrac{1}{9} \\ y = 3\sqrt{3} \end{array}\right\}$ or $\left.\begin{array}{l} x = 27 \\ y = \dfrac{1}{3} \end{array}\right\}$

4. $f(x)=3x^3+Ax^2-4x+B$

$$f\left(-\frac{2}{3}\right)=-\frac{8}{9}+\frac{4A}{9}+\frac{8}{3}+B=0$$

$4A+9B=-16$ ——— (1)

$f(-1)=-3+A+4+B=2$

$A+B=1$ ——— (2)

From (1) and (2) $A=5,\ B=-4$

$$3x^3+5x^2-4x-4=(3x+2)(x^2+x-2)$$
$$=(3x+2)(x+2)(x-1)$$

5. $f(x)=x^4+hx^3+gx^2-16x-12$

$f(-1)=1-h+g+16-12=0$

$h-g=5$ ——— (1)

$f(1)=1+h+g-16-12=-24$

$h+g=3$ ——— (2)

From (1) and (2) $h=4,\ g=-1$

$f(x)=x^4+4x^3-x^2-16x-12$

$f(2)=16+32-4-32-12=0$

$(x-2)$ is a factor of $f(x)$

$f(-1)=1-4-1+16-12=0$

$(x+1)$ is a factor of $f(x)$

$$x^4+4x^3-x^2-16x-12=(x+1)(x^3+3x^2-4x-12)$$
$$=(x+1)(x-2)(x^2+5x+6)$$
$$=(x+1)(x-2)(x+2)(x+3)$$

6. $ax^2+bx+c=0$

$$\alpha+\beta=-\frac{b}{a},\quad \alpha\beta=\frac{c}{a}$$

$$x+2+\frac{1}{x}=\frac{b^2}{ac}$$

$$x^2-\left(\frac{b^2}{ac}-2\right)x+1=0$$

$$x^2-\left(\frac{(\alpha+\beta)^2}{\alpha\beta}-2\right)x+1=0$$

$$x^2-\left(\frac{\alpha}{\beta}+\frac{\beta}{\alpha}\right)x+1=0$$

$$\left(x-\frac{\alpha}{\beta}\right)\left(x-\frac{\beta}{\alpha}\right)=0$$

$$x=\frac{\alpha}{\beta} \text{ or } \frac{\beta}{\alpha}$$

7. Let α be the common root of the equations $x^2+bx+ca=0,\ x^2+cx+ab=0$

Then $\alpha^2+b\alpha+ca=0$ ——— (1)

$\alpha^2+c\alpha+ab=0$ ——— (2)

(1) - (2) gives $\alpha=\frac{a(b-c)}{(b-c)}=a$

If α,β are the roots of the Equations சமன்பாடு (1),

$\alpha\beta=ca$ and $\alpha=a$ Implies that $\beta=c$

If α,γ are the roots of the Equations சமன்பாடு (2),

$\alpha\gamma=ab$ and $\alpha=a$ Implies that $\gamma=b$

$\alpha+\beta=-b$ என்பதால் $a+c=-b$

The equation whose roots are β and γ is

$$(x-\beta)(x-\gamma)=0$$
$$x^2-(\beta+\gamma)x+\beta\gamma=0$$
$$x^2-(b+c)x+bc=0$$
$$x^2+ax+bc=0$$

8. $g(x)=ax^2-2x+(3a+2)$

$g(x)$ to be positive for all real values of x

$a>0$ and $\Delta<0$

$a>0$ and $4-4a(3a+2)<0$

$$3a^2+2a-1>0$$
$$(3a-1)(a+1)>0$$
$$a<-1 \text{ or } a>\frac{1}{3}$$

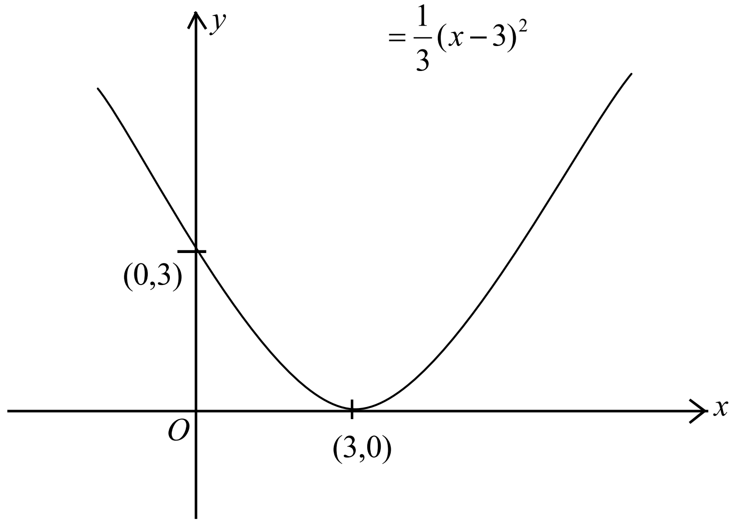
$a > 0$ என்பதால், $a > \frac{1}{3}$

Solution: $\left\{x : x \in \mathbb{R}, x > \frac{1}{3}\right\}$

When $a = \frac{1}{3}$, $g(x) = \frac{1}{3}x^2 - 2x + 3$

$$= \frac{1}{3}\left[x^2 - 6x + 9\right]$$

$$= \frac{1}{3}(x-3)^2$$



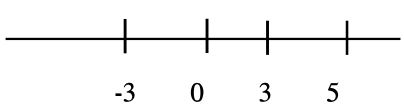
09. $\frac{12}{x-3} \le x+1$

$$\frac{12}{x-3} - (x+1) \le 0$$

$$-\frac{(x^2 - 2x - 15)}{x-3} \le 0$$

$$-\frac{(x-5)(x+3)}{x-3} \le 0$$

$(x-5)(x+3)(x-3) \ge 0 (x \ne 3)$



$-3 \le x < 3$ or $x \ge 5$

10. $|1-2x|-|x+2|\le 2$

When $x<-2$, $1-2x+(x+2)\le 2$

$x \ge 1$ no solution ——— (1)

When $-2\le x<\dfrac{1}{2}$, $1-2x-(x+2)\le 2$

$-3x-1\le 2$

$x\ge -1$

Solution is $-1\le x<\dfrac{1}{2}$ ——— (2)

When $x\ge\dfrac{1}{2}$, $-(1-2x)-(x+2)\le 2$

$-1+2x-x-2\le 2$

$x\le 5$

$\dfrac{1}{2}\le x\le 5$ ——— (3)

$-1\le x\le 5$

Hence, solution is $=\{x : x\in R, -1\le x\le 5\}$

11. 8 children can sit ins 8! ways

$8! = 40\ 320$

(i) The number of ways that the two particular girls can sit together is $2\times 7!$

Hence the number of ways the two particular girls do not si together is

$8!-2\times 7!$

$=7!(8-2)$

$=7!\times 6=30240$

(ii) 4 boys can be arranged in 4! ways $=4!$

$\uparrow \overset{*}{B_1} \uparrow \overset{*}{B_2} \uparrow \overset{*}{B_3} \uparrow \overset{*}{B_4} \uparrow$

4 girls can be seated in $5\times 4\times 3\times 2=5!$

Hence number of ways

$=4!\times 5!=2880$

12. $\left(x^2 - \frac{2k}{x}\right)^{10}$

$$T_{r+1} = {}^{10}C_r \left(x^2\right)^{10-r} \left(-\frac{2k}{x}\right)^r$$

$$= {}^{10}C_r (-2k)^r x^{20-3r}$$

Coefficient of x^2: $20 - 3r = 2$

$$r = 6$$

Coefficient of x^2 : ${}^{10}C_6 (-2k)^6$

Coefficient of x^{-1}: $20 - 3r = -1$

$$r = 7$$

Coefficient of x^{-1} is ${}^{10}C_7 (-2k)^7$

$${}^{10}C_6 (-2k)^6 = {}^{10}C_7 (-2k)^7$$

$$\frac{10!}{6! \times 4!}(-2k)^6 = \frac{10!}{7! \times 3!}(-2k)^7$$

$$k = -\frac{7}{8}$$

13. $(1 + 2x + kx^2)^n$

$$= \left[1 + x(2 + kx)\right]^n$$

$$= 1 + nC_1 x(2 + kx) + {}^nC_2 x^2 (2 + kx)^2 + {}^nC_3 x^3 (2 + kx)^3 + \ldots$$

Coefficient of x^2 : $k.{}^nC_1 + 4.{}^nC_2$

$$= nk + 2n(n-1)$$

Coefficient of x^3 : $4k.{}^nC_2 + 8.{}^nC_3$

$$= 2n(n-1)k + \frac{4n(n-1)(n-2)}{3}$$

$$nk + 2n(n-1) = 30 \quad \text{———} (1)$$

$$2n(n-1)k + \frac{4n(n-1)(n-2)}{3} = 0 \quad \text{———} (2)$$

From (2) $k + \frac{2(n-2)}{3} = 0$

Substituting in (1), $\dfrac{-2n(n-2)}{3}+2n(n-1)=30$

$2n^2-n-45=0$

$(2n+9)(n-5)=0$

$n=5$, since n is positive in teger

$n=5$ and $k=-2$

14. $Z=-1+i\sqrt{3}$

$=2\left(-\dfrac{1}{2}+i\dfrac{\sqrt{3}}{2}\right)=2\left(\cos\dfrac{2\pi}{3}+i\sin\dfrac{2\pi}{3}\right)$

$|Z|=2, Arg(Z)=\dfrac{2\pi}{3}$

$Z^2=\left(-1+i\sqrt{3}\right)^2=-2-i2\sqrt{3}$

$Z^2+pz=\left(-2-i2\sqrt{3}\right)+p\left(-1-i\sqrt{3}\right)$

$=(-2-p)+i\left(\sqrt{3}p-2\sqrt{3}\right)$

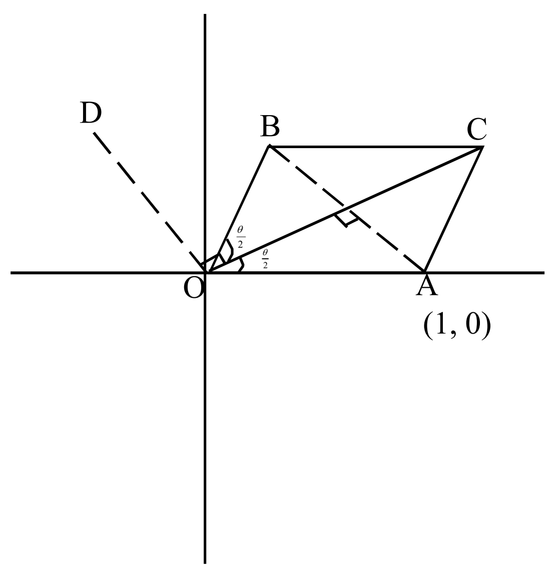
Z^2+pz real, $\sqrt{3}p-2\sqrt{3}=0;\quad p=2$

$Z^2+qz=(-2-q)+i\left(\sqrt{3}q-2\sqrt{3}\right)$

$\dfrac{\sqrt{3}(q-2)}{-(q+2)}=\tan\dfrac{5\pi}{6}=-\dfrac{1}{\sqrt{3}}$

$q=4$

15. $OA=|z|=1$



$OB=|\cos\theta+i\sin\theta|=1$

$OACB$ a parallelogram

Point C, represents Z_1+Z_2

Since $OA=OB$, $OACB$ is rhombus

$OD=AB$, OD is parallel to AB

Point D, represents Z_2-Z_1

$|Z_1+Z_2|=OC=2\cos\dfrac{\theta}{2}$

$Arg(Z_1+Z_2)=\dfrac{\theta}{2}$

$$AB = |Z_2 - Z_1| = 2\sin\frac{\theta}{2}$$

$$Arg(Z_2 - Z_1) = \frac{\pi}{2} + \frac{\theta}{2}$$

$$|Z_1 + Z_2|^2 + |Z_2 - Z_1|^2$$

$$= \left(2\cos\frac{\theta}{2}\right)^2 + \left(2\sin\frac{\theta}{2}\right)^2 = 4$$

16. (a) $\lim\limits_{x \to a} \dfrac{\sin x - \sin a}{x - a}$

$$= \lim_{x \to a} \frac{2\cos\left(\frac{x+a)}{2}\right)\sin\left(\frac{x-a)}{2}\right)}{2 \times \left(\frac{x-a)}{2}\right)}$$

$$= \lim_{x \to a} \cos\left(\frac{x+a)}{2}\right) \times \frac{\sin\left(\frac{x-a)}{2}\right)}{\left(\frac{x-a)}{2}\right)}$$

$$= \cos a$$

(b) $\sin y = x.\sin(y+a)$ ——————— (1)

Differentiating w.r.t x

$$\cos y.\frac{dy}{dx} = x.\cos(y+a).\frac{dy}{dx} + \sin(y+a)$$

From (1) $\quad x = \dfrac{\sin y}{\sin(y+a)}$

$$\cos y.\frac{dy}{dx} = \frac{\sin y}{\sin(y+a)}.\cos(y+a).\frac{dy}{dx} + \sin(y+a)$$

$$\left[\cos y - \frac{\sin y.\cos(y+a)}{\sin(y+a)}\right]\frac{dy}{dx} = \sin(y+a)$$

$$\frac{\sin a}{\sin(y+a)}.\frac{dy}{dx} = \sin(y+a)$$

$$\frac{dy}{dx} = \frac{\sin^2(y+a)}{\sin a}$$

17. (a) $\lim\limits_{x\to 0}\dfrac{\tan x-\sin x}{x^3}$

$$=\lim_{x\to 0}\frac{\sin x(1-\cos x)}{\cos x.x^3}$$
$$=\lim_{x\to 0}\frac{\sin x}{x}\times\frac{1-\cos x}{x^2}\times\frac{1}{\cos x}$$
$$=\lim_{x\to 0}\frac{\sin x}{x}\times\frac{2\sin^2\frac{x}{2}}{4\times\left(\frac{x}{2}\right)^2}\times\frac{1}{\cos x}=1\times\frac{1}{2}\times 1=\frac{1}{2}$$

(b) $y=x^n.lnx$

$$\frac{dy}{dx}=x^n.\frac{1}{x}+lnx.n.x^{n-1}$$
$$x.\frac{dy}{dx}=x^n+n.lnx.x^n$$
$$x.\frac{dy}{dx}=x^n+ny$$
$$x.\frac{d^2y}{dx^2}+\frac{dy}{dx}=n.x^{n-1}+n.\frac{dy}{dx}$$
$$x.\frac{d^2y}{dx^2}+(1-n)\frac{dy}{dx}=n.x^{n-1}$$

Hence $n=3$

18. $x=t+lm\,\mathrm{t}$ $\qquad$ $y=t-lm\,\mathrm{t}$

$$\frac{dx}{dt}=1+\frac{1}{t}=\frac{t+1}{t}\qquad\qquad \frac{dy}{dt}=1-\frac{1}{t}=\frac{t-1}{t}$$

$$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{t-1}{t+1}\text{ ——————— }(1)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dt}\left(\frac{dy}{dx}\right) \times \frac{dt}{dx}$$

$$= \frac{d}{dt}\left(\frac{t-1}{t+1}\right) \times \frac{t}{t+1}$$

$$= \frac{2t}{(t+1)^3}$$

Since $t = \dfrac{x+y}{2}$

$$\frac{d^2y}{dx^2} = \frac{x+y}{\left(\frac{x+y}{2}+1\right)^3} = \frac{8(x+y)}{(x+y+2)^3}$$

19. $\dfrac{1}{1+x^2} - \dfrac{1}{(1+x)^2}$

$$= \frac{(1+x)^2 - (1+x^2)}{(1+x^2)(1+x)^2}$$

$$= \frac{2x}{(1+x^2)(1+x)^2}$$

$$\int_0^1 \frac{x}{(1+x^2)(1+x)^2} dx = \frac{1}{2}\left[\int_0^1 \frac{1}{1+x^2}\, dx - \int \frac{1}{(1+x)^2}\, dx\right]$$

$$= \frac{1}{2}\left[\tan^{-1} x + \frac{1}{1+x}\right]_0^1$$

$$= \frac{1}{2}\left[\left(\tan^{-1} 1 + \frac{1}{2}\right) - (0+1)\right]$$

$$= \frac{1}{2}\left[\tan^{-1} 1 - \frac{1}{2}\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{4} - \frac{1}{2}\right] = \frac{\pi}{8} - \frac{1}{4}$$

20. $x = 2(1+\cos^2\theta)$

$x \to 2 \qquad \theta \to \frac{\pi}{2}$

$x \to 3 \qquad \theta \to \frac{\pi}{4}$

$\frac{dx}{d\theta} = -4\cos\theta.\sin\theta$

$\int_2^3 \sqrt{\frac{x-2}{4-x}}dx$

$= \int_{\frac{\pi}{2}}^{\frac{\pi}{4}} \sqrt{\frac{2\cos^2\theta}{2\sin^2\theta}}.(-4\cos\theta.\sin\theta.d\theta)$

$= \int_{\frac{\pi}{2}}^{\frac{\pi}{4}} \frac{\cos\theta}{\sin\theta}.(-4\cos\theta.\sin\theta)d\theta$

$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 4\cos^2\theta d\theta = 2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (1+\cos 2\theta)d\theta$

$= 2\left[\theta + \frac{\sin 2\theta}{2}\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$

$= 2\left[\left(\frac{\pi}{2}+0\right)-\left(\frac{\pi}{4}+\frac{1}{2}\right)\right]$

$= \frac{\pi}{2} - 1$

21. $I = \int e^{4x}.\cos 3x dx, \qquad J = \int e^{4x}.\sin 3x.dx$

$I = \int e^{4x}.\cos 3x dx = e^{4x}.\frac{\sin 3x}{3} - \int \frac{\sin 3x}{3} \times 4e^{4x} dx$

$3I + 4J = e^{4x}.\sin 3x$ ——— (1)

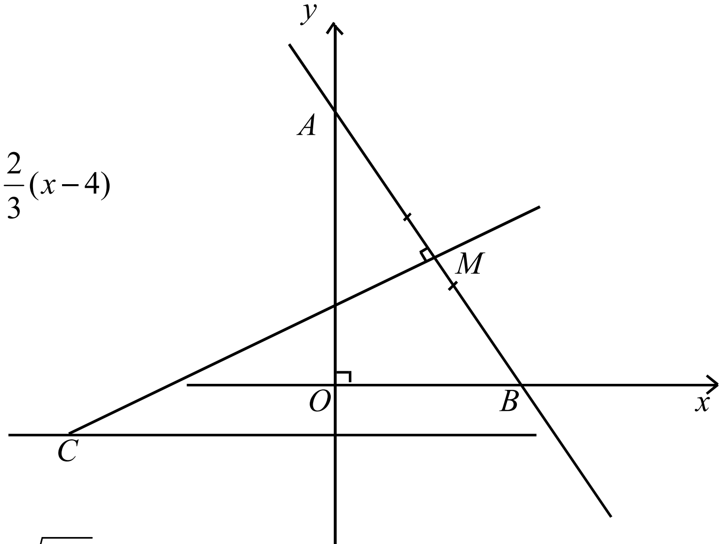
$J = \int e^{4x}.\sin 3x dx = e^{4x}.\left(\frac{-\cos 3x}{3}\right) - \int\left(\frac{-\cos 3x}{3}\right) \times 4e^{4x} dx$

$4I + 3J = e^{4x}.\cos 3x$ ——— (2)

From (1) and (2) $\qquad I = \frac{1}{25}e^{4x}(3\sin 3x + 4\cos 3x)$

22. $A \equiv (0,12)$, $B \equiv (8,0)$

$M \equiv (4,6)$



Equation of MC is $y-6=\frac{2}{3}(x-4)$

$3y-2x-10=0$

At C, $y=-1$, $x=-\frac{13}{2}$

$C \equiv (-\frac{13}{2},-1)$

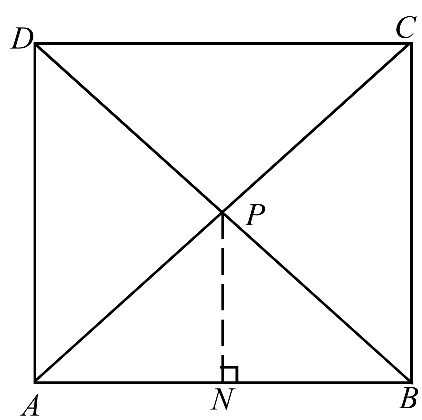
Distance AB $=\sqrt{8^2+12^2}=\sqrt{208}$

Distance MC $=\sqrt{(4+13)^2+(6+1)^2}=\sqrt{\frac{441}{4}+49}$

Area of the tri angle $=\frac{1}{2}\times\sqrt{208}\times\sqrt{\frac{637}{4}}=\frac{364}{4}$

$=$ 91 square units.

23. Equation of AB: $x-2y=0$



$P \equiv \left(\frac{5}{2},\frac{5}{2}\right)$

PN is perpendicular to AB and

$$PN=\frac{\left|\frac{5}{2}-5\right|}{\sqrt{5}}=\frac{\sqrt{5}}{2}$$

Since DC is parallel to AB

Equation of DC is $x-2y+k=0$.

and the pependicular distance from P to CD is $\frac{5}{2}$

$$\frac{\left|\frac{5}{2}-5+k\right|}{\sqrt{5}}=\frac{\sqrt{5}}{2}$$

$|2k-5|=5$, $k=5$ or 0

Hence equation of CD is $x-2y+5=0$

BC and AD are perpendicular to $x-2y=0$

Equation of BC and AD are of the form $2x+y+d=0$

and the perpendicular distance from P $\dfrac{\sqrt{5}}{2}$

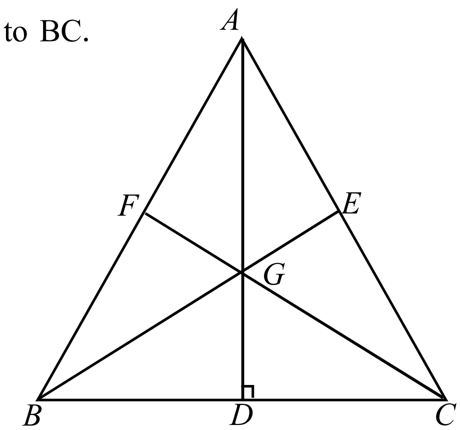
$$\frac{\left|2\times\frac{5}{2}+\frac{5}{2}+d\right|}{\sqrt{5}}=\frac{\sqrt{5}}{2}$$

$d=-10,-5$

Hence the equations are $2x+y-5=0,\quad 2x+y-10=0$

24. It is given that AB = AC. So AD is pependicular to BC.

All three medians meet at a point G.



$A\equiv(0,8)$

$BE: x+3y=14$

$CF: 3x-y=2$

$G\equiv(2,4),\quad D\equiv(x_0,y_0)$

$AG:GC=2:1$

$$\frac{2x_0+0}{2+1}=2,\quad \frac{2y_0+8}{2+1}=4$$

$D\equiv(x_0,y_0)\equiv(3,2)$

Equation of BC is $y-2=\dfrac{1}{2}(x-3)$

$2y-x-1=0$

$$\left.\begin{aligned}BC&: 2y-x-1=0\\ BE&: 3y+x-14=0\end{aligned}\right\}\quad B\equiv(5,3)$$

$$\left.\begin{aligned}BC&: 2y-x-1=0\\ CF&: 3x-y-2=0\end{aligned}\right\}\quad C\equiv(1,1)$$

Equation of AB is $y-3=-1(x-5)$

$y+x-8=0$

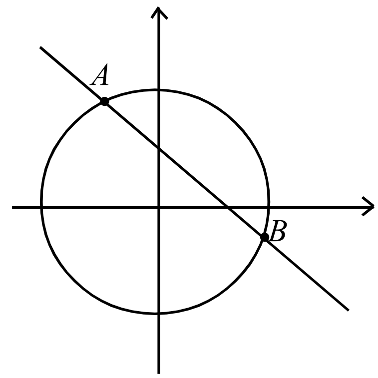
Equation of AC is $y-1=-7(x-1)$

$y+7x-8=0$

25. $S \equiv x^2 + y^2 - a^2 = 0$

$l \equiv x\cos\alpha + y\sin\alpha - p = 0$

Any circle passing through the points A and B can be written



$$\left(x^2 + y^2 - a^2\right) + \lambda\left(x\cos\alpha + y\sin\alpha - p\right) = 0$$

center is $\left(-\dfrac{\lambda\cos\alpha}{2}, -\dfrac{\lambda\sin\alpha}{2}\right)$

AB is the diameta of the circle

$\left(-\dfrac{\lambda\cos\alpha}{2}, -\dfrac{\lambda\sin\alpha}{2}\right)$, lies on $x\cos\alpha + y\sin\alpha - p = 0$

$$-\frac{\lambda\cos\alpha}{2}.\cos\alpha - \frac{\lambda\sin\alpha}{2}.\sin\alpha - p = 0$$

$\lambda = 2p$

Equation of the requird circle is

$$\left(x^2 + y^2 - a^2\right) - 2p\left(x\cos\alpha + y\sin\alpha - p\right) = 0$$

26. Centre $C \equiv (2,1)$

$P \equiv (4,2)$

Radius $= \sqrt{4+1-4} = 1$

$CP = \sqrt{(4-2)^2 + (2-1)^2} = \sqrt{5}$

(1) $CP > 1$, P lies outside S

(11) $PT = \sqrt{CP^2 - 1^2} = \sqrt{5-1} = 2$

Suppose that the equation of tangent is $y = mx + c$

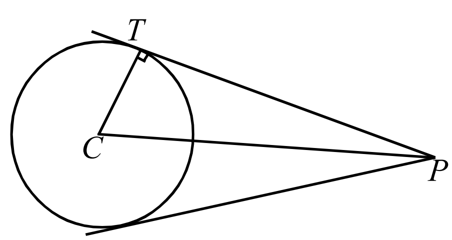
It passes through *P(4, 2)*

$2 = 4m + c$

$y = mx + (2 - 4m)$

$y - mx - (2 - 4m = 0$

$CT = 1$



$$\frac{|1 - 2m - 2 + 4m|}{\sqrt{1+m^2}} = 1$$

$|2m - 1| = \sqrt{1+m^2}$

$(2m-1)^2 = m^2 + 1$

$m = 0$ or $\frac{4}{3}$

If $m = 0$, $C = 2$

If $m = \frac{4}{3}$, $C = -\frac{10}{3}$

Equatons of the tangentsare $y = 2$, and $3y - 4x + 10 = 0$

27. Equation of the circle is $x^2 + y^2 + 2gx + 2fy + c = 0$

Centre $(-g, -f)$ and radius is equal to $\sqrt{g^2 + f^2 - c}$

Perpendicular distance from C to y axis is equal to the radius of the circle.

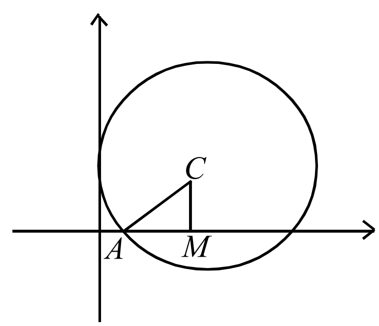
Equation of y axis is $x = 0$

$$\frac{|g|}{1} = \sqrt{g^2 + f^2 - c}$$

$$g^2 = g^2 + f^2 - c$$

$$c = f^2$$

Equation: $x^2 + y^2 + 2gx + 2fy + f^2 = 0$



$$AC^2 = AM^2 + MC^2$$

$$g^2 + f^2 - f^2 = \left(\frac{3}{2}\right)^2 + f^2$$

$$g^2 = f^2 + \frac{9}{4}$$

Therefore, the general equation is

$$x^2 + y^2 + 2\left(\sqrt{f^2 + \frac{9}{4}}\right)x + 2fy + f^2 = 0$$

Centre $\left(-\sqrt{f^2 + \frac{9}{4}}, -f\right)$

$$x_0 = -\sqrt{f^2 + \frac{9}{4}}, \qquad y_0 = -f$$

$$x_0^2 - y_0^2 = \frac{9}{4}$$

$$4x_0^2 - 4y_0^2 = 9$$

Locus of (x_0, y_0) is $4x^2 - 4y^2 = 9$

28. $\cos 6\theta + \cos 4\theta + \cos 2\theta + 1 = 0 \qquad (o < \theta < \pi)$

$2\cos 5\theta.\cos\theta + 2\cos^2\theta = 0$

$2\cos\theta(\cos 5\theta + \cos\theta) = 0$

$4\cos\theta.\cos 3\theta.\cos 2\theta = 0$

$\cos\theta = 0 \qquad \cos 3\theta = 0 \qquad \cos 2\theta = 0$

$\theta = 2n\pi \pm \frac{\pi}{2} \qquad 3\theta = 2n\pi \pm \frac{\pi}{2} \qquad 2\theta = 2n\pi \pm \frac{\pi}{2}; n \in \mathbb{Z}$

$\theta = \frac{\pi}{2}; \qquad \theta = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}; \qquad \theta = \frac{\pi}{4}, \frac{3\pi}{4}$

$$\theta = \left\{\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{5\pi}{6}\right\}$$

29. $\tan^{-1}\left(\frac{1}{3}\right) = A$

$\tan A = \frac{1}{3},$

$\tan 2A = \frac{2\tan A}{1 - \tan^2 A} = \frac{3}{4}$

$0 < 2A < \frac{\pi}{4}, \qquad 2A = \tan^{-1}\frac{3}{4}$

Let $\tan^{-1}\left(\frac{1}{7}\right) = B$

$\tan B = \frac{1}{7}$ and $0 < B < \frac{\pi}{4}$

$2\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right)$

$= 2A + B$ and $0 < 2A + B < \frac{\pi}{2}$

$$\tan(2A + B) = \frac{\tan 2A + \tan B}{1 - \tan 2A.\tan B} = \frac{\frac{3}{4} + \frac{1}{7}}{1 - \frac{3}{4} \times \frac{1}{7}} = 1$$

$2A + B = \frac{\pi}{4}$

$2\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right) = \frac{\pi}{4}$

30. $$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=\frac{b+c-a}{\sin B+\sin C-\sin A}$$

$$\frac{a}{\sin A}=\frac{b+c-a}{\sin B+\sin C-\sin A}$$

$$\frac{a}{2\sin\frac{A}{2}\cos\frac{A}{2}}=\frac{b+c-a}{2\sin\left(\frac{B+C}{2}\right)+\cos\left(\frac{B-C}{2}\right)-2\sin\frac{A}{2}\cos\frac{A}{2}}$$

$$\frac{a}{\sin\frac{A}{2}}=\frac{b+c-a}{\cos\left(\frac{B-C}{2}\right)-\sin\frac{A}{2}}$$

$$\frac{a}{\sin\frac{A}{2}}=\frac{b+c-a}{\cos\left(\frac{B-C}{2}\right)-\cos\left(\frac{B+C}{2}\right)}$$

$$\frac{a}{\sin\frac{A}{2}}=\frac{b+c-a}{2\sin\frac{B}{2}.\sin\frac{C}{2}}$$

$$\frac{a}{\sin\frac{A}{2}}.\cos\frac{A}{2}=\frac{b+c-a}{2\sin\frac{B}{2}.\sin\frac{C}{2}}.\cos\frac{A}{2}$$

$$2a\cot\frac{A}{2}=(b+c-a)\frac{\sin\left(\frac{B+C}{2}\right)}{\sin\frac{B}{2}.\sin\frac{C}{2}}$$

$$2a\cot\frac{A}{2}=(b+c-a)\frac{\sin\frac{B}{2}.\cos\frac{C}{2}+\cos\frac{B}{2}.\sin\frac{C}{2}}{\sin\frac{B}{2}.\sin\frac{C}{2}}$$

$$2a\cot\frac{A}{2}=(b+c-a)\left(\cot\frac{C}{2}+\cot\frac{B}{2}\right)$$

31. Let $\mathbb{Z} = r(\cos\theta + i\sin\theta)$ for all $n \in \mathbb{Z}^+$, $\mathbb{Z}^n = r^n(\cos n\theta + i\sin n\theta)$.

$\mathbb{Z} = 1 + \sqrt{3}i$

$= \mathbb{Z}\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$

$= \mathbb{Z}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

Demovier's theorem,

$$\mathbb{Z}^7 = 2^7\left(\cos\frac{7\pi}{3} + i\sin\frac{7\pi}{3}\right)$$

$$= 128\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

$$\left|\mathbb{Z}^7\right| = 128$$

$$Arg(\mathbb{Z})^7 = \frac{\pi}{3}$$

32. Let $\mathbb{Z} = r(\cos\theta + i\sin\theta)$ for all $n \in \mathbb{Z}^+$, $\mathbb{Z}^n = r^n(\cos n\theta + i\sin n\theta)$.

If $\mathbb{Z} = \cos\theta + i\sin\theta$ then

$\mathbb{Z}^3 = (\cos\theta + i\sin\theta)^3 = (\cos 3\theta + i\sin 3\theta)$

$\cos^3\theta + 3\cos^2\theta(i\sin\theta) + 3\cos\theta(i\sin\theta)^2 + (2\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$

$(\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\cos^2\theta\sin\theta - \sin^3\theta) = \cos 3\theta + i\sin 3\theta$

Equating the real part

$\cos^3\theta - 3\cos\theta\sin^2\theta = \cos 3\theta$

$\cos^3\theta - 3\cos\theta(1 - \cos^2\theta) = \cos 3\theta$

$\cos^3\theta - 3\cos\theta + 3\cos^3\theta = \cos 3\theta$

$\therefore \cos 3\theta = 4\cos^3\theta - 3\cos\theta$

By equating the imaginary part

$3\cos^2\theta\sin\theta - \sin^3\theta = \sin 3\theta$

$3(1-\sin^2\theta)\sin\theta - \sin^3\theta = \sin 3\theta$

$\sin^3\theta - 3\sin\theta + 3\sin^3\theta = \sin 3\theta$

$\therefore \sin 3\theta = 3\sin\theta - 4\sin^3\theta$

33. $y = t^2(1-t)$

$\frac{dy}{dx} = t^2(-1) + (1-t)2t = -t^2 + 2t - 3t^2$

$= -3t^2 + 2t$

$= t(2-3t)$

$x = t(1-t)^2$

$\frac{dx}{dy} = t.2(1-t)(-1) + (1-t)^2.1$

$= -2t + 2t^2 + t^2 + 1 - 2t$

$= 3t^2 - 4t + 1$

$= (1-t)(1-3t)$

$\frac{dx}{dy} = \frac{t(2-3t)}{(1-t)(1-3t)}$

$t = T$

$\frac{dx}{dy}\Big| t = T = \frac{t(2-3t)}{(1-t)(1-3t)}$

$\frac{dx}{dy} = \frac{\frac{1}{4}}{-\frac{1}{4}} = -1$

$y = \frac{1}{4}\left(1-\frac{1}{2}\right) = \frac{1}{8}$

$x = \frac{1}{2}\left(1-\frac{1}{2}\right)^2 = \frac{1}{8}$

$$\frac{y-\frac{1}{8}}{x-\frac{1}{8}}=-1$$

$$\frac{8y-1}{8x-1}=-1$$

$$8y-1=-8x+1$$

$$8y+8x-2=0$$

$$4y+4x-1=0$$

34. $y=x^2-3x$

$$y=x^2-3x+\frac{9}{4}-\frac{9}{4}$$

$$y=\left(x-\frac{3}{2}\right)^2-\frac{9}{4}$$

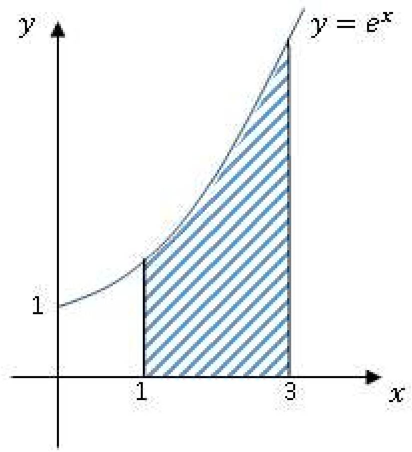
$$\left|\int_0^3 (x^2-3x)dx\right|$$

$$=\left|\left[\frac{x^3}{3}-\frac{3x^2}{2}\right]_0^3\right|$$

$$=\left|\frac{27}{3}-\frac{27}{2}\right|$$

$$=\left|\frac{27}{6}\right| \quad =\frac{27}{6} \text{ square units}$$

(35)



$$R = \int_1^3 e^x dx$$

$$= \left[e^x \right]_1^3$$

$$= e^3 - e^1 \quad = e(e^2 - 1)$$

$$V = \int_1^3 \pi y^2 dx$$

$$= \int_1^3 \pi (e^x)^2 dx$$

$$= \int_1^3 \pi e^{2x} dx \quad = \pi \left[\frac{e^{2x}}{2} \right]_1^3$$

$$= \pi \left[\frac{e^6}{2} - \frac{e^2}{2} \right]$$

$$= \frac{\pi}{2} e^2 (e^4 - 1)$$

$$V = \frac{\pi e^2}{2} (e^4 - 1)$$

Part B

1. (a) $x^2 + px + q = 0$

$\alpha + \beta = -p, \quad \alpha\beta = q$

(i) $|\alpha - \beta| = 2\sqrt{3}$

$$\frac{1}{\alpha} + \frac{1}{\beta} = 4$$

$$\frac{\alpha + \beta}{\alpha\beta} = 4$$

$$-p = 4q$$

$$(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$$

$$12 = p^2 - 4q$$

$$p^2 + p - 12 = 0$$

$$(p+4)(p-3) = 0$$

$$\left.\begin{array}{l} p = -4 \\ q = 1 \end{array}\right\} \qquad \left.\begin{array}{l} p = 3 \\ q = -\frac{3}{4} \end{array}\right\}$$

(ii) $\alpha + \dfrac{2}{\beta} = \dfrac{\alpha\beta + 2}{\beta} = \dfrac{q+2}{\beta}$

$$\beta + \frac{2}{\alpha} = \frac{\alpha\beta + 2}{\alpha} = \frac{q+2}{\alpha}$$

$$x^2 + px + q = 0 \quad \text{——————} \quad (1)$$

Let $y = \dfrac{q+2}{x}$

$$x = \frac{q+2}{y}$$

Replacing x by $\dfrac{q+2}{y}$ in equation (1),

we have

$$\left(\frac{q+2}{y}\right)^2 + p\left(\frac{q+2}{y}\right) + q = 0$$

$$(q+2)^2 + p(q+2)y + qy^2 = 0$$

$$qy^2 + p(q+2)y + (q+2)^2 = 0$$

i.e The equation whose roots are $\alpha + \dfrac{2}{\beta}, \beta + \dfrac{2}{\alpha}$ is

$$qx^2 + p(q+2)x + (q+2)^2 = 0$$

(b) Let $y = \dfrac{x^2 + 3x - 4}{5x - k}$

$$x^2 + (3 - 5y)x + (ky - 4) = 0$$
$$\Delta = (3 - 5y)^2 - 4(ky - 4)$$
$$= 25y^2 - (4k + 30)y + 25$$

For real values of x, $\Delta \geq 0$

i.e $25y^2 - (4k + 30)y + 25 \geq 0$

For all values of y, $25y^2 - (4k + 30)y + 25$ to be greaten than or equal to zero

(i) Coefficent of $y^2 = 25 > 0$ and

(ii) $\Delta_1 = (4k + 30)^2 - 4 \times 25 \times 25 \leq 0$

$$(4k + 30)^2 - 50^2 \leq 0$$
$$(4k - 20)(4k + 80) \leq 0$$
$$(k - 5)(k + 20) \leq 0$$
$$-20 \leq k \leq 5$$

k=-5

$$f(x) = \frac{(x+4)(x-1)}{5(x+1)}$$

(1) When $x = 0$, $f(x) = -\dfrac{4}{5}$

(2) When $y = 0$, -4, 1

(3) x = -1 is an assymptot.

(4) $f(x) = -\dfrac{(x+4)\left(1 - \frac{1}{x}\right)}{5\left(1 + \frac{1}{x}\right)}$ $\quad f(x) \longrightarrow \infty,\ x \longrightarrow \infty$ as

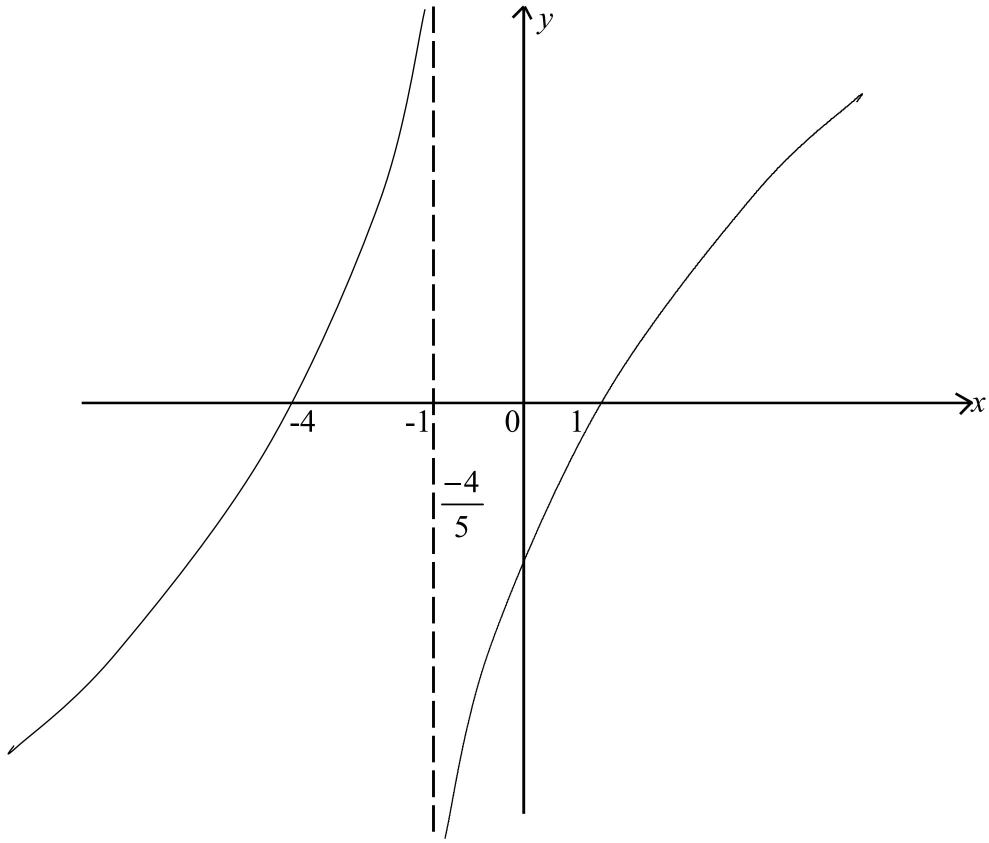
$f(x) \longrightarrow -\infty,\ x \longrightarrow -\infty$ as

(5) $x < -4$, $\quad f(x) < 0$

$-4 < x < -1$, $\quad f(x) > 0$

$-1 < x < 1$, $\quad f(x) < 0$

$x > 1$, $\quad f(x) > 0$



02. $f(x)=\lambda^2x^2-(\lambda^2-2\lambda)x+3=0$

$$\alpha+\beta=\frac{\lambda^2-2\lambda}{\lambda^2}$$

$$\alpha\beta=\frac{3}{\lambda^2}$$

$$\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{\alpha^2+\beta^2}{\alpha\beta}=\frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}=\frac{4}{3}$$

$$\frac{\left(\frac{\lambda^2-2\lambda}{\lambda^2}\right)^2-2\times\frac{3}{\lambda^2}}{\frac{3}{\lambda^2}}=\frac{4}{3}$$

$$\frac{(\lambda^2-2\lambda)^2-6\lambda^2}{3\lambda^2}=\frac{4}{3}$$

$$3\lambda^4-12\lambda^3+12\lambda^2-18\lambda^2=12\lambda^2$$
$$3\lambda^4-12\lambda^3-18\lambda^2=0$$

$$\lambda^2\left(\lambda^2-4\lambda-6\right)=0$$
$$\lambda^2-4\lambda-6=0$$
$$\lambda_1+\lambda_2=4$$
$$\lambda_1\lambda_2=-6$$

Equaton whose roots are $\dfrac{\lambda_1^2}{\lambda_2}$, $\dfrac{\lambda_2^2}{\lambda_1}$

$$x^2-\left[\frac{\lambda_1^2}{\lambda_2}+\frac{\lambda_2^2}{\lambda_1}\right]x+\lambda_1\lambda_2=0$$
$$x^2-\left[\frac{\left(\lambda_1^3+\lambda_2^3\right)}{\lambda_1\lambda_2}\right]x+\lambda_1\lambda_2=0$$
$$x^2-\left[\frac{(\lambda_1+\lambda_2)\left(\lambda_1^2-\lambda_1\lambda_2+\lambda_2^2\right)}{\lambda_1\lambda_2}\right]x+\lambda_1\lambda_2=0$$
$$x^2-\left[\frac{4[16+18]}{-6}\right]x-6=0$$
$$3x^2+68x-18=0$$

If $f(x)>2\lambda x$ then $f(x)-2\lambda x>0$ for all $x\in\mathbb{R}$

$$\lambda^2x^2-\lambda^2x+3>0$$
$$x^2-x+\frac{3}{\lambda^2}>0$$
$$x^2-x+\frac{1}{4}-\frac{1}{4}+\frac{3}{\lambda^2}>0$$
$$\left(x-\frac{1}{2}\right)^2+\left(\frac{12-\lambda^2}{4\lambda^2}\right)>0$$

Since $\left(x-\dfrac{1}{2}\right)^2\geq 0$ for all $x\in\mathbb{R}$

$$\frac{12-\lambda^2}{4\lambda^2}\geq 0$$
$$12-\lambda^2>0$$

$\lambda^2 - 12 \leq 0$

$\left(\lambda + 2\sqrt{3}\right)\left(\lambda - 2\sqrt{3}\right) \leq 0$

$-2\sqrt{3} \leq \lambda \leq 2\sqrt{3}$

$-3.42 \leq \lambda \leq 3.42$

Greatest integer value of λ is 3

(b) $\displaystyle\sum_{r=1}^{2n} (-1)^{r+1} \frac{1}{r} = \sum_{r=n+1}^{2n} \frac{1}{r}$

When $n=1$, L.H.S $= \displaystyle\sum_{r=1}^{2} (-1)^{r+1} \frac{1}{r}$

$= 1 - \frac{1}{2} = \frac{1}{2}$

R.H.S $= \displaystyle\sum_{r=2}^{2} \frac{1}{r} = \frac{1}{2}$

L.H.S = R.H.S

The result is true for $n=1$

Assume that the result is true for $n = p$

$$\sum_{r=1}^{2p} (-1)^{r+1} \frac{1}{r} = \sum_{r=p+1}^{2p} \frac{1}{r}$$

i.e $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} ... - \frac{1}{2p} = \frac{1}{p+1} + \frac{1}{p+2} + ... + \frac{1}{2p}$

$n = p+1$ as, $\displaystyle\sum_{r=1}^{2(p+1)} (-1)^{r+1} \frac{1}{r} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} ... - \frac{1}{2p} + \frac{1}{2p+1} - \frac{1}{2p+2}$

$$= \left(\frac{1}{p+1} + \frac{1}{p+2} + ... + \frac{1}{2p}\right) + \frac{1}{2p+1} - \frac{1}{2p+2}$$

$$= \frac{1}{p+2} + \frac{1}{p+3} + ... + \frac{1}{2p} + \frac{1}{2p+1} - \frac{1}{2p+2} + \frac{1}{p+1}$$

$$= \frac{1}{p+2} + \frac{1}{p+3} + ... + \frac{1}{2p} + \frac{1}{2p+1} + \frac{1}{2p+2}$$

$$= \sum_{r=p+2}^{2(p+1)} \frac{1}{r}$$

The result is true for $n = p+1$

By the principle of mathematic induction, the result is true for all positive intergern.

03. (a) $$\frac{2r+3}{r(r+1)}=\frac{A}{r}+\frac{B}{r+1}$$

$$=\frac{A(r+1)+Br}{r(r+1)}$$

$$=\frac{(A+B)r+A}{r(r+1)}$$

$$2r+3=(A+B)r+A$$

$A=3,\ B=-1$ $$\frac{2r+3}{r(r+1)}=\frac{3}{r}=\frac{1}{r+1}$$

$$Ur=\frac{2r+3}{r(r+1)}\times\frac{1}{3^r}$$

$$=\left[\frac{3}{r}-\frac{1}{r+1}\right].\frac{1}{3^r}$$

$$=\left[\frac{1}{r}.\frac{1}{3^{r-1}}-\frac{1}{r+1}.\frac{1}{3^r}\right]=V_r-V_{r+1}$$

$$V_r=\frac{1}{r.3^{r-1}}$$

$$\underline{U_r=V_r-V_{r+1}}$$

$$u_1=v_1-v_2$$

$$u_2=v_2-v_3$$

$$u_3=v_3-v_4$$

....

$$u_{n-1}=v_{n-1}-v_n$$

$$\underline{u_n=v_n-v_{n+1}}$$

$$\sum_{r=1}^{n}U_r=V_1-V_{n+1}$$

$$\sum_{r=1}^{n}U_r=\frac{1}{1}-\frac{1}{n+1}.\frac{1}{3^n}$$

$n\to\alpha$ ஆக $\frac{1}{n+1}.\frac{1}{3^n}\to 0$

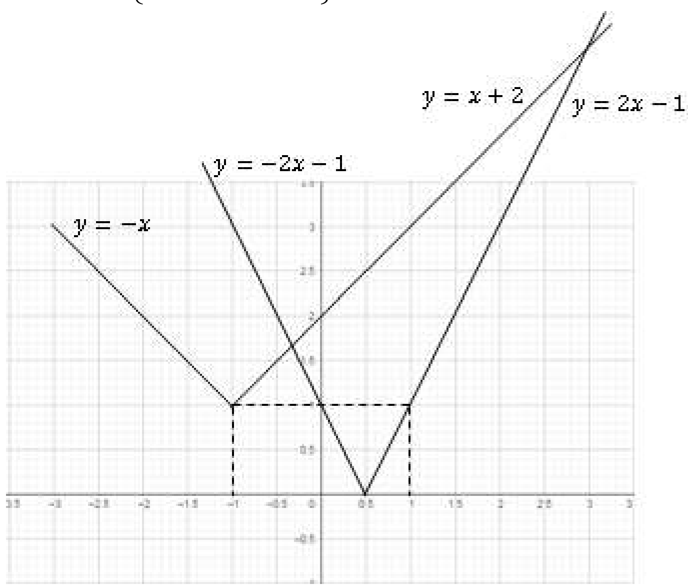
$$\lim_{n\to\alpha}\sum_{r=1}^{n}U_r=1$$

Hence the series is convergent and $\sum_{r=1}^{\alpha}U_n=1$

(b)

$$y = |2x-1| = \begin{cases} 2x-1, & x \geq \dfrac{1}{2} \\ -2x+1, & x < \dfrac{1}{2} \end{cases}$$

$$y = |x+1|+1 = \begin{cases} x+2, & x \geq -1 \\ -x, & x < -1 \end{cases}$$



$$\begin{aligned} y &= x+2 \\ y &= -2x+1 \\ x+2 &= -2x+1 \\ 3x &= -1 \\ x &= -\frac{1}{3} \end{aligned}$$

$$\begin{aligned} y &= x+2 \\ y &= 2x-1 \\ x+2 &= 2x-1 \\ x &= 3 \end{aligned}$$

$$|2x-1| - |x+1| \geq 1$$

$$|2x-1| \geq 1 + |x+1|$$

$\therefore$ solution $x \geq 3$ and $x \leq -\dfrac{1}{3}$

04.(a)(i) Consider the six girls as one group.

Now 7 can be arranged in a row in 7! ways.

6 girls can be arrange among them selves in 6! ways.

Hence the total number of ways that the six girls.

can sit together is $7! \times 6!$! ways.

$$= 5040 \times 720$$

$$= 3628800$$

(ii) G G G G G G ———————— (1)

$\uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow$

G G G G G G ———————— (2)

$\uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow$

Six girls can sit in 6! ways.

Six boys can be arranged as shown above in two ways.

In each way, boys can be arranged in 6! ways.

Hence the number of ways the boys and girls sit alternatively is $2\times 6!\times 6!$

$= 2\times 720\times 720$

$= 1036800$

(a) 0, 2, 3, 5, 7, 8

$\boxed{* \quad * \quad * \quad *}$

(i) $=5\times6\times6\times6=1080$ numbers can be formed.

(ii) One digit only once

$=5\times5\times4\times3=300$ numbers can be formed.

Greater than 5000 and divisible by 2

* * * 0 ——— ends in zero, $3\times4\times3\times1=36$

2 ——— ends in 2, $3\times4\times3\times1=36$

8 ——— ends in 8, $2\times4\times3\times1=24$

Total $=36+36+24=96$

(c) $(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \ldots + {}^nC_rx^r + \ldots + {}^nC_nx^n$

$(x+1)^n = {}^nC_0x^n + {}^nC_1x^{n-1} + {}^nC_2x^{n-2} + \ldots + + {}^nC_rx^{n-r} + \ldots + {}^nC_n$

differentiating w.r.t x

(1) $n(1+x)^{n-1} = {}^nC_1 + 2.{}^nC_2x + \ldots + r.{}^nC_rx^{r-1} + \ldots + n.{}^nC_nx^{n-1}$

(2) $n(x+1)^{n-1} = n.{}^nC_0x^{n-1} + (n-1){}^nC_1x^{n-2} + \ldots + (n-r){}^nC_rx^{n-r-1} + 1.{}^nC_{n-1}$

Consider $(1)\times(2)$

$n^2(1+x)^{2n-2} = \left({}^nC_1 + \ldots + n.{}^nC_nx^{n-1}\right)\left(n.{}^nC_0x^{n-1} + \ldots + C_{n-1}\right)$

Coefficient of x^{n-2}

In R.H.S, coefficient x^{n-2} is

$$(n-1)^n C_1^2 + 2(n-2)^n C_2^2 + ... + r(n-r)^n C_{r1}^2 + ... + (n-1)^n C_n^2$$

In L.H.S, coefficient of x^{n-2} is $n^2 . {}^{2n-2}C_{n-2}$

Hence the result.

In (3) put $x=1$.

$$n^2.2^{2n-2} = \left({}^nC_1 + 2.{}^nC2_1 + ... + n.{}^nC_n\right)\left(n.{}^nC_0 + ... + 1.{}^nC_{n-1}\right)$$

$$n^2.2^{2n-2} = \sum_{r=1}^{n} r.{}^nC_r . \sum_{r=0}^{n-1}(n-r).{}^nC_r$$

05. (a) $Z^3 = 1$

$$\left(Z-1\right)\left(Z^2+Z+1\right)=0$$

$Z-1=0$ or $Z^2+Z+1=0$

$Z=1$

$$Z\frac{-1\pm\sqrt{-3}}{2}$$

$Z=1$ or $-\frac{1}{2}+i\frac{\sqrt{3}}{2}$ or $-\frac{1}{2}-i\frac{\sqrt{3}}{2}$

Let ω be a comlex root of $Z^3-1=0$

Now

$$\omega^3 - 1 = 0$$

$$\left(\omega-1\right)\left(\omega^2+\omega+1\right)=0$$

$\omega \neq 1$ Therefore $1+\omega+\omega^2=0$

(i) $1+\omega = -\omega^2$

$$\frac{1}{1+\omega} = -\frac{1}{\omega^2}$$

$$\frac{\omega}{1+\omega} = -\frac{1}{\omega}$$

(ii) $1+\omega^2 = -\omega$

$$\frac{1}{1+\omega^2} = -\frac{1}{\omega}$$

$$\frac{\omega^2}{\omega^2+1} = -\omega$$

(iii) $\left(\dfrac{\omega}{1+\omega}\right)^{3k}+\left(-\dfrac{\omega^2}{1+\omega}\right)^{3k}$

$$=\left(-\frac{1}{\omega}\right)^{3k}+(-\omega)^{3k}$$

$$=(-1)^{3k}\left[\frac{1}{\left(\omega^3\right)^k}+\left(\omega^3\right)^k\right]$$

$$=(-1)^{3k}\left[1+1\right]$$

$$=(-1)^{3k}.2$$

If k odd, $(-1)^{3k}.2=-2$

If k even, $(-1)^{3k}.2=2$

(b) $u=2i=2\left(\cos\dfrac{\pi}{2}+i\sin\dfrac{\pi}{2}\right)$

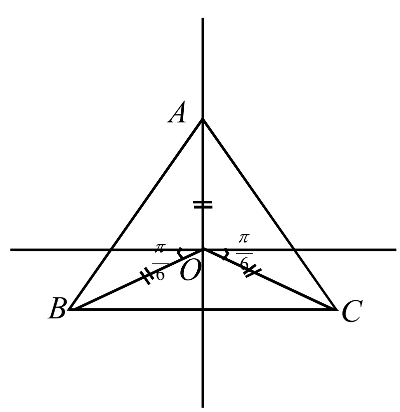
$$v=-\frac{1}{2}+i\frac{\sqrt{3}}{2}=1\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$$

$$uv=2\left(\cos\left(\frac{\pi}{2}+\frac{2\pi}{3}\right)+i\sin\left(\frac{\pi}{2}+\frac{2\pi}{3}\right)\right)=2\left(\cos\frac{7\pi}{6}+i\sin\frac{7\pi}{6}\right)$$

$$=2\left(\cos\left(-\frac{5\pi}{6}\right)+i\sin\left(-\frac{5\pi}{6}\right)\right)$$

$$\frac{u}{v}=2\left(\cos\left(\frac{\pi}{2}-\frac{2\pi}{3}\right)+i\sin\left(\frac{\pi}{2}-\frac{2\pi}{3}\right)\right)$$

$$=2\left(\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)\right)$$



$OA=OB=OC$

It can be easily proved that

$B\hat{A}C=A\hat{B}C=A\hat{C}B=60^0$

Hence ABC is an equilateral triangle.

06. (*a*) $\left(\frac{1+i}{1-i}\right)^{4n+1}$

$$=\left(\frac{1+i}{1-i}\times\frac{1+i}{1+i}\right)^{4n+1}$$

$$=\left(\frac{2i}{2}\right)^{4n+1}=i^{4n+1}=\left(i^4\right)^n i=i$$

$$x^3-1=0$$

$$(x-1)\left(x^2+x+1\right)=0$$

$$x=1,\qquad x^2+x+1=0, x=\frac{-1\pm\sqrt{-3}}{2}$$

$$x=1,\qquad x=-\frac{1}{2}+i\frac{\sqrt{3}}{2},\qquad x=-\frac{1}{2}-i\frac{\sqrt{3}}{2}$$

$$x=1,\qquad x=\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3},\quad x=\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}$$

$$x=1,\qquad \omega,\omega^2$$

Also, $1+\omega+\omega^2=0,\quad \omega^3=1$

$(x+2)^3=1;\quad y^3=1;\quad y=1,\omega,\omega^2$

$x+2=y$

$$x+2=1,\quad x+2=-\frac{1}{2}+i\frac{\sqrt{3}}{2},\quad x+2=-\frac{1}{2}-i\frac{\sqrt{3}}{2}$$

$$x=-1,\quad x=-\frac{5}{2}+i\frac{\sqrt{3}}{2},\quad x=-\frac{5}{2}-i\frac{\sqrt{3}}{2}$$

$$\left(2+5\omega+2\omega^2\right)^6=\left(2+2\omega+2\omega^2+3\omega\right)^6$$

$$=(3\omega)^6=3^6.\omega^6=729$$

$$(p-q)(p\omega-q)\left(p\omega^2-q\right)$$

$$=(p-q)\left[p^2\omega^3-pq\omega^2-pq\omega+q^2\right]$$

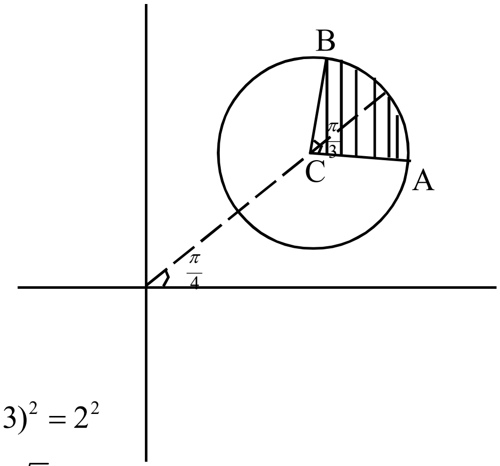
$$=(p-q)\left(p^2+pq+q^2\right)=p^3-q^3$$

$$\omega\left(b+c\omega+a\omega^2\right)=b\omega+c\omega^2+a\omega^3$$

$$=a+b\omega+c\omega^2$$

$$\frac{a+b\omega+c\omega^2}{b+c\omega+a\omega^2}=\omega$$

(b)



$|Z-3-3i|=2$

Locus of P is a circle with centre (3, 3) and radius 2.

Equation of the locus is $(x-3)^2+(y-3)^2=2^2$

The treatest value of $|Z|$ in the region is $3\sqrt{2}+2$

07. (a) $\lim\limits_{x\to 0}\dfrac{\cos 4x-\cos^2 x}{x^2}$

$$=\lim_{x\to 0}\frac{1-2\sin^2 2x-\cos^2 x}{x^2}$$

$$=\lim_{x\to 0}\frac{\sin^2 x}{x^2}-2.\frac{\sin^2 2x}{x^2}$$

$$=\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2-2\times 4\times\frac{\sin^2 2x}{(2x)^2}$$

$$=1-8\times 1$$

$$=-7$$

$$\lim_{x\to 0}\frac{\tan 2x-2\sin x}{x^3}$$

$$=\lim_{x\to 0}\frac{\dfrac{\sin 2x}{\cos 2x}-2\sin x}{x^3}$$

$$=\lim_{x\to 0}\frac{2\sin x}{x}\left(\frac{\cos x-\cos 2x}{x^2}\right)\times\frac{1}{\cos 2x}$$

$$=\lim_{x\to 0}\frac{2\sin x}{x}\times\frac{2\sin\frac{3x}{2}}{x}\times\frac{\sin\frac{x}{2}}{x}\times\frac{1}{\cos 2x}$$

$$=\lim_{x\to 0}\frac{2\sin x}{x}\times\frac{2\sin\frac{3}{2}x}{\frac{3x}{2}}\times\frac{3}{2}\times\frac{\sin\frac{x}{2}}{2\times\frac{x}{2}}\times\frac{1}{\cos 2x}$$

$$=2\times 2\times\frac{3}{2}\times\frac{1}{2}\times 1=3$$

(*b*) (i) $y=\sin^{-1}\dfrac{1}{\sqrt{x^2-1}}$, $\qquad Z=\sec^{-1}x \qquad \left(x>\sqrt{2}\right)$

$\sin y=\dfrac{1}{\sqrt{x^2-1}} \qquad x=\sec z$

$$\sin y=\frac{1}{\sqrt{\sec z^2-1}}=\frac{1}{\sqrt{\tan^2 z}}=\cot z$$

$$\cos y.\frac{dy}{dz}=-\cos ec^2 z$$

Since $x>\sqrt{2}, \quad 0<y<\dfrac{\pi}{2}, \quad 0<z<\dfrac{\pi}{2}$

$$\sqrt{1-\sin^2 y}.\frac{dy}{dz}=-\left(1+\cot^2 z\right)$$

$$\sqrt{1-\frac{1}{x^2-1}}.\frac{dy}{dz}=-\left(1+\frac{1}{\tan^2 z}\right)$$

$$\sqrt{\frac{x^2-2}{x^2-1}}.\frac{dy}{dz}=-\frac{\sec^2 z}{\sec^2 z-1}=-\frac{x^2}{x^2-1}$$

$$\frac{dy}{dz}=-\frac{x^2}{x^2-1}\times\sqrt{\frac{x^2-1}{x^2-2}}$$

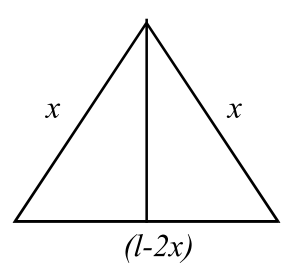
$$\frac{dy}{dz}=\frac{-x^2}{\sqrt{\left(x^2-2\right)\left(x^2-1\right)}}$$

$$\frac{dy}{dz}+\frac{x^2}{\sqrt{\left(x^2-2\right)\left(x^2-1\right)}}=0$$

(ii) $\dfrac{dy}{dz}+\dfrac{x^2}{\sqrt{\left(x^2-1\right)\left(x^2-2\right)}}=0$

(*c*) Area $A=\left(\dfrac{l}{2}-x\right)\sqrt{x^2-\left(\dfrac{l}{2}-x\right)^2}$

$$=\left(\frac{l}{2}-x\right)\sqrt{lx-\frac{l^2}{4}}$$



$$\frac{dA}{dx}=\left(\frac{l}{2}-x\right)\times\frac{1}{2}\times\frac{1}{\sqrt{lx-\frac{l^2}{4}}}\times l+\sqrt{lx-\frac{l^2}{4}}\,(-1)$$

$$=\frac{\frac{l}{2}\left(\frac{l}{2}-x\right)-\left(lx-\frac{l^2}{4}\right)}{\sqrt{lx-\frac{l^2}{4}}}$$

$$=\frac{\frac{l^2}{2}-\frac{3lx}{2}}{\sqrt{lx-\frac{l^2}{4}}}$$

$$=\frac{-3l}{2\sqrt{lx-\frac{l^2}{4}}}\left(x-\frac{l}{3}\right)$$

$\frac{l}{4}<x<\frac{l}{3},\quad \frac{dA}{dx}>0$ A increases.

$x>\frac{l}{3},\quad \frac{dA}{dx}<0$ A decreases.

Hence A has a maximum at $x=\frac{l}{3}$ and the triangle is equitlateral triange.

$$\text{Area }=\frac{1}{2}\times\frac{l}{3}\times\frac{l}{3}\times\sin 60$$

$$=\frac{1}{2}\times\frac{l}{3}\times\frac{l}{3}\times\frac{\sqrt{3}}{2}=\frac{\sqrt{3}l^2}{36}\text{ sq.units.}$$

08. (*a*) (i) $f(x)=\sin 2x$

$$f'(x)=\lim_{h\to 0}\frac{\sin(2x+2h)-\sin 2x}{h}$$

$$=\lim_{h\to 0}\frac{2\cos(2x+h)\cdot\sin h}{h}$$

$$=\lim_{h\to 0}2\cos(2x+2h)\frac{\sin h}{h}$$

$$=2\cos 2x.\times 1=2\cos 2x$$

(ii) $\dfrac{d^n}{dx^n}\left(\sin 2x\right) = 2^n \sin\left[\dfrac{n\pi}{2} - 2x\right]$

when $n=1$

L.H.S. $= \dfrac{d}{dx}(\sin 2x) = 2\cos 2x$

R.H.S. $= 2sin\left(\dfrac{\pi}{2} - 2x\right) = 2\cos 2x$

The result is true for $n=1$.

Assume that the result is true for $n = p$

$$\frac{d^p}{dx^p}(\sin 2x) = 2^p.\sin\left(\frac{p\pi}{2} - 2x\right)$$

$$\frac{d^{p+1}}{dx^{p+1}}(\sin 2x) = \frac{d}{dx}\left[2^p.\sin\left(\frac{p\pi}{2} - 2x\right)\right]$$

$$= 2^p.\cos\left(\frac{p\pi}{2} - 2x\right)\times(-2)$$

$$= 2^{p+1}\left[-\cos\left(\frac{p\pi}{2} - 2x\right)\right]$$

$$= 2^{p+1}.\sin\left[\frac{\pi}{2} + \left(\frac{p\pi}{2} - 2x\right)\right]$$

$$= 2^{p+1}.\sin\left[(p+1)\frac{\pi}{2} - 2x\right]$$

Therefore the result is true for $n = p+1$

By the principle of mathematical induction the result is true for n all positive intergern.

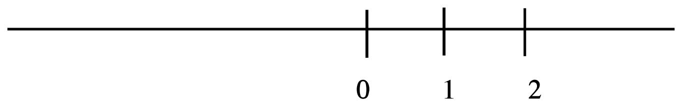
(b) $f(x) = 1 + \dfrac{1}{x(x-2)}$

$$f'(x) = \frac{-(2x-2)}{x^2(x-2)^2}$$

$$= \frac{-2(x-1)}{x^2(x-2)^2}$$

When $x=1$, $f'(x)=0$

$x=0$ and $x=2$ are asymptotes.



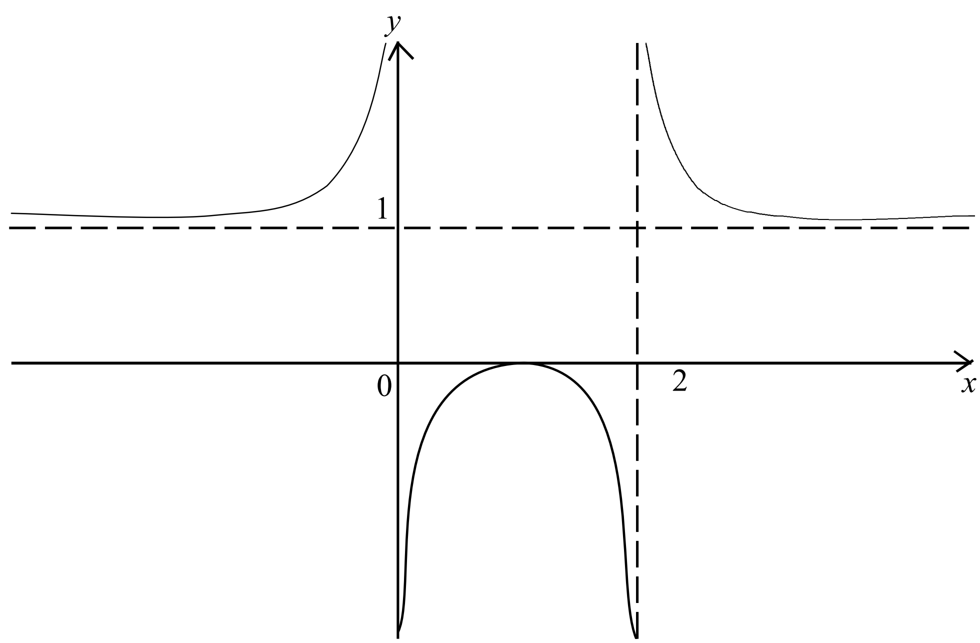
$x<0$	$f'(x)>0$	f is increasing.
$0<x<1$	$f'(x)>0$	f is increasing.
$i<x<2$	$f'(x)<0$	f is decreasing.
$x>2$	$f'(x)<0$	f is decreasing.

At $x=1$, f has a maximum and $f(1)=0$.

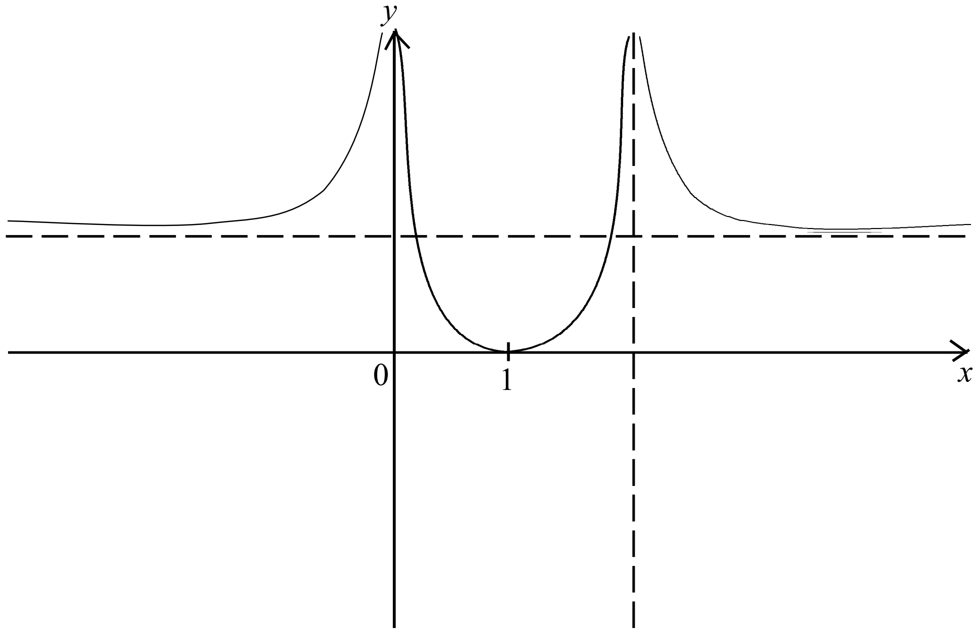
$f(x)\to 1$ as $x\to\pm\infty$

$y=1$ is an asymptot.

(i) $y=f(x)$



(ii) $y = |f(x)|$



$$f(x) = 1 + \frac{1}{x(x-2)}$$

$$= \frac{(x-1)^2}{x(x-2)}$$

$$\frac{1}{f(x)} = \frac{x(x-2)}{(x-1)^2}$$

$f(x) \to 1$ as $x \to \pm\alpha$; $\frac{1}{f(x)} \to 1$ as $x \to \pm\infty$

At $x = 0,\ 2,\ \frac{1}{f(x)} = 0$

When $x < 0$ $\frac{1}{f(x)}$ is increasing.

$0 < x < 1$ $\frac{1}{f(x)}$ is increasing.

Thereforce $x > 1$ $\frac{1}{f(x)}$ is decreasing.

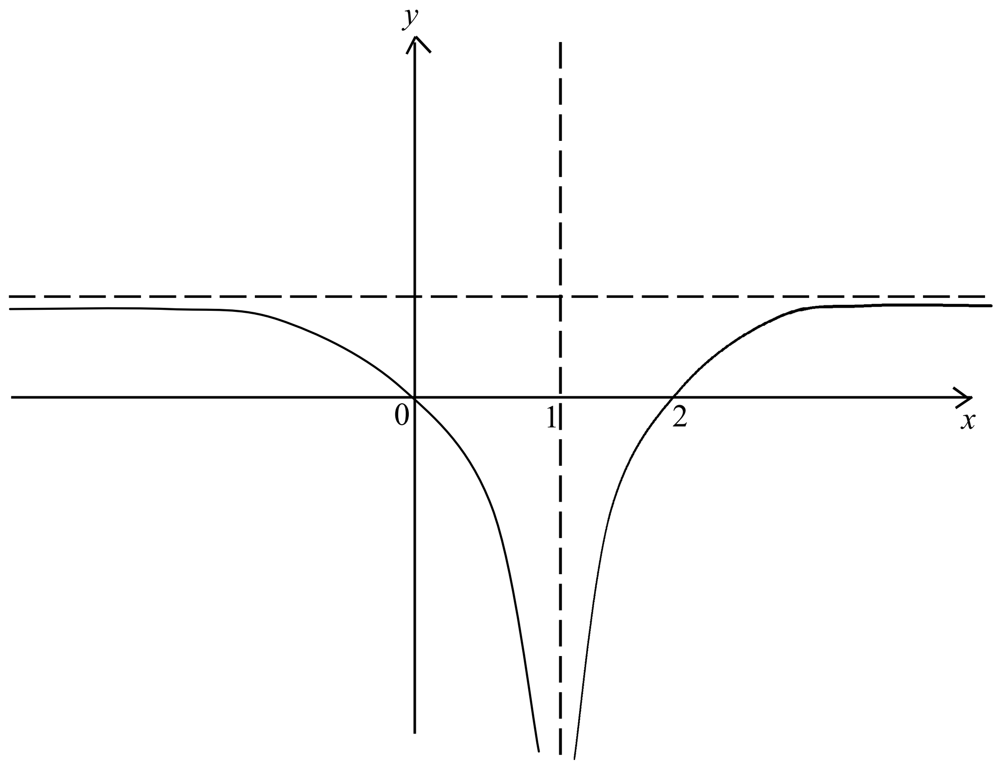
$f(x) = 0$

At $x = 1$, $y = \frac{1}{f(x)}$

Therefore is an asymptot of $y=\dfrac{1}{f(x)}$

Since $1<x<2$, and $x>2$, $f(x)$ decreases.

$\dfrac{1}{f(x)}$ increases.



09. (a) $\dfrac{1}{\left(1-x^2\right)\left(x^2+1\right)}=\dfrac{A}{1+x}+\dfrac{B}{1-x}+\dfrac{Cx+D}{1+x^2}$

$1=A(1+x^2)(1-x)+B(1+x)(1+x^2)+(Cx+D)(1+x)(1-x)$

$x=1,\quad 1=4B,\quad B=\dfrac{1}{4}$

$x=-1,\quad 1=4A,\quad A=\dfrac{1}{4}$

$x=0,\quad 1=A+B+D,\quad D=\dfrac{1}{2}$

Coefficient of x^3 $\quad 0=-A+B-C,\quad C=0$

$$\int\frac{1}{\left(1-x^2\right)\left(1+x^2\right)}dx=\int\frac{1}{4(1+x)}dx+\int\frac{1}{4}\frac{1}{(1-x)}dx+\frac{1}{2}\int\frac{dx}{1+x^2}$$

$$=\frac{1}{4}\ln|1+x|-\frac{1}{4}\ln|1-x|+\frac{1}{2}\tan^{-1}x+c$$

(b) $t = \sin x - \cos x$

$$t^2 = (\sin x - \cos x)^2 = 1 - \sin 2x$$

$$\sin 2x = 1 - t^2$$

$$t = \sin x - \cos x \qquad x: 0 \to \frac{\pi}{4}$$

$$\frac{dt}{dx} = \cos x + \sin x \qquad t: -1 \to 0$$

$$\int_0^{\frac{\pi}{4}} \frac{\sin x + \cos x}{9 + 16\sin 2x} dx = \int_{-1}^{0} \frac{dt}{9 + 16(1 - t^2)}$$

$$= \int_{-1}^{0} \frac{dt}{(5-4t)(5+4t)}$$

$$= \int_{-1}^{0} \left\{ \frac{A}{(5-4t)} + \frac{B}{(5+4t)} \right\} dt$$

$$5A + 5B = 1$$

$$4A - 4B = 0$$

$$A = \frac{1}{10}, B = \frac{1}{10}$$

$$= \frac{1}{10} \int_{-1}^{0} \frac{dt}{(5-4t)} + \frac{1}{10} \int_{-1}^{0} \frac{dt}{(5+4t)}$$

$$= \frac{-1}{40} \ln|5-4t| + \frac{1}{40} \ln|5+4t|$$

$$= \frac{1}{40} \left[\ln\left|\frac{5+4t}{5-4t}\right| \right]_{-1}^{0}$$

$$= \frac{1}{40} \left[\ln 1 - \ln\frac{1}{9} \right]$$

$$= \frac{1}{40} \ln 9$$

(c) $I = \int_0^{\frac{\pi}{2}} \frac{\cos x}{a\cos x + b\sin x} dx, \quad J = \int_0^{\frac{\pi}{2}} \frac{\sin x dx}{a\cos x + b\sin x} dx$

$$aI + bJ = \int_0^{\frac{\pi}{2}} \frac{a\cos x + b\sin x}{a\cos x + b\sin x} dx = [x]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$$

$$bI - aJ = \int_0^{\frac{\pi}{2}} \frac{b\cos x - a\sin x}{a\cos x + b\sin x} dx$$

$$= \left[\ln\left|a\cos x + b\sin x\right|\right]_0^{\frac{\pi}{2}}$$

$$= \ln\left|\frac{b}{a}\right|$$

$$I = \frac{1}{a^2 + b^2}\left[\frac{\pi a}{2} + bln\left|\frac{b}{a}\right|\right]$$

$$J = \frac{1}{a^2 + b^2}\left[\frac{\pi b}{2} - aln\left|\frac{b}{a}\right|\right]$$

10. (a) $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x} dx = \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1+\cos^2(\pi - x)} dx = \int_0^{\pi} \frac{(\pi - x)\sin x}{1+\cos^2 x} dx$

$$= \int_0^{\pi} \frac{\pi \sin x dx}{1+\cos^2 x} - \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x} dx$$

$$2\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x} dx = \pi \int_0^{\pi} \frac{\sin x}{1+\cos^2 x} dx$$

Put $u = \cos x$

$$\frac{du}{dx} = -\sin x$$

$x : 0 \to \pi$

$u : 1 \to -1$

$$\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x} dx = \frac{\pi}{2} \int_0^{\pi} \frac{\sin x}{1+\cos^2 x} dx$$

$$= \frac{\pi}{2} \int_1^{-1} \frac{-du}{1+u^2}$$

$$= \frac{\pi}{2} \int_{-1}^{+1} \frac{+du}{1+u^2}$$

$$=\frac{\pi}{2}\left[\tan^{-1}u\right]_{-1}^{1}$$

$$=\frac{\pi}{2}\left[\tan^{-1}(1)-\tan^{-1}(-1)\right]$$

$$=\frac{\pi}{2}\left[\frac{\pi}{4}-\left(-\frac{\pi}{4}\right)\right]=\frac{\pi^2}{4}$$

(b) $\int\frac{x.e^x}{(1+x)^2}dx$

$$=\int x.e^x.\frac{1}{(1+x)^2}\,dx$$

$u=x.e^x$ $\qquad \frac{dv}{dx}=\frac{1}{(1+x)^2}$

$$v=\frac{-1}{(1+x)}$$

$$\int\frac{x.e^x}{(1+x)^2}dx=x.e^x\,\frac{-1}{(1+x)}-\int\frac{-1}{(1+x)}.e^x(x+1)dx$$

$$=\frac{-x.e^x}{1+x}+\int e^x\,dx$$

$$=\frac{-x.e^x}{1+x}+e^x$$

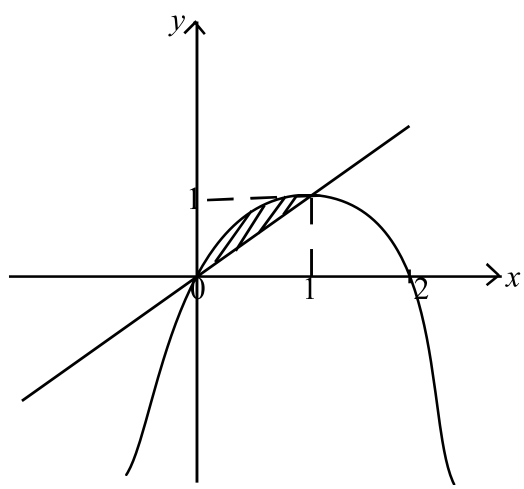
$$=\frac{e^x}{1+x}$$

(c) $y=x(2-x)$

$$=-\left(x^2-2x\right)$$

$$=-\left[x^2-2x+1-1\right]$$

$$y=-(x-1)^2+1$$



$$x(2-x)=x$$
$$x(2-x)-x=0$$
$$x(1-x)=0$$
$$x=0,1$$

$$\text{Area } = \int_0^1 \left(2x - x^2\right)dx - \int_0^1 x\,dx$$

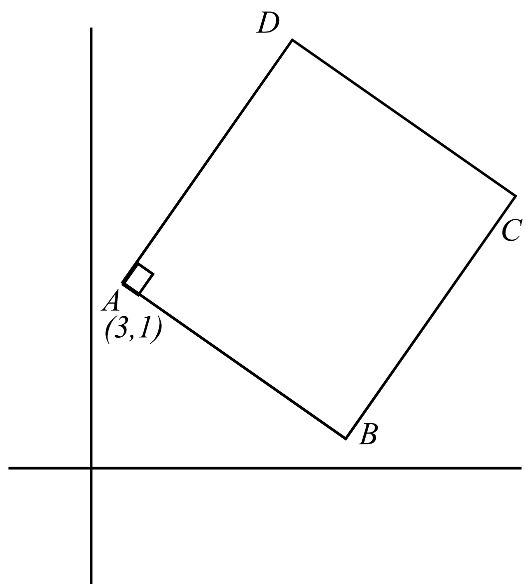
$$= \int_0^1 \left(x - x^2\right)dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$

$$= \frac{1}{6} \text{ sq. units.}$$

11. (*a*) AD : $x + y - 4 = 0$

AC : $3x - y - 8 = 0$

$A \equiv (3,1)$



Equaton of AB is

$(y + x - 4) + \lambda(y - 3x + 8) = 0$

$(1 - 3\lambda)x + (1 + \lambda)y + (8\lambda - 4) = 0$

Gradient of AB $= \dfrac{3\lambda - 1}{\lambda + 1}$

Gradien of AD is $= -1$

Gradient of AB is $= 1 = \dfrac{3\lambda - 1}{\lambda + 1}$

$\lambda = 1$

Equaton of AB is $= x - y - 2 = 0$

Let $B \equiv (x_0, y_0)$

$$\frac{y_0 - 1}{x_0 - 3} = 1$$

$$\frac{y_0 - 1}{1} = \frac{x_0 - 3}{1} \;(= t \text{ , say})$$

$$\left(x_0 - 3\right)^2 + \left(y_0 - 1\right)^2 = \left(2\sqrt{2}\right)^2$$

$t^2 + t^2 = 8$

$2t^2 = 8$

$t^2 = 4$

$t = \pm 2$

if $t = 2$, $B \equiv (5,3)$

$t = -2$, $B \equiv (+1, -1)$

Since B lies in the first quadrant

$B \equiv (5,3)$

Equation of BC is $y + x = k \quad (AD // BC)$

$$5 + 3 = k$$
$$k = 8$$

Equation of BC is $y + x = 8$

$BD,\ x - 3y + 7 = 0$

Equation of BD is $= x - 3y + c = 0$

$$5 - 9 + c = 0$$
$$c = 4$$

Equation of BD is $x - 3y + 4 = 0$

BD : $x - 3y + 4 = 0$

AD : $x + y - 4 = 0$

$D \equiv (2,2)$

Equation of CD is $x - y = k$

$$2 - 2 = k$$
$$k = 0$$

Equation of CD is $x - y = 0$

(b) $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$

S: $(2,0), (0,-1)$

$4 + 0 + 4g + 0 + c = 0$

$0 + 1 + 0 - 2f + c = 0$

$$4 + 4g + c = 0$$
$$1 - 2f + c = 0$$

$$g = \frac{-(c+4)}{4}, \quad f = \frac{c+1}{2}$$

$$S \equiv x^2 + y^2 - \frac{2(c+4)}{4}x + \frac{2(c+1)}{2}y + c = 0$$
$$2x^2 + 2y^2 - (c+4)x + 2(c+1)y + 2c = 0$$

The general equation of the circle is

$$S \equiv x^2 + y^2 - \left(\frac{\lambda+4}{2}\right)x + (\lambda+1)y + \lambda = 0$$ ஆகும்.

Since the circle passes through (1, -1)

$$1+1-\left(\frac{\lambda+4}{2}\right)-(\lambda+1)+\lambda=0$$

$$\lambda=-2$$

(i) Equation of S_1 is $x^2+y^2-x-y-2=0$

centre $C_1 \equiv \left(\frac{1}{2},\frac{1}{2}\right)$

(ii) $S_1=0$ bisects $S_2 \equiv x^2+y^2-\left(\frac{\lambda+4}{2}\right)x+(\lambda+1)y+\lambda=0$

Common chord of $S_1=0$ and $S_2=0$ is $S_1-S_2=0$

$$\left(\frac{\lambda+4}{2}-1\right)x-(\lambda+2)y-(\lambda+2)=0$$

$$(\lambda+2)x-2(\lambda+2)y-2(\lambda+2)=0$$

Common chord passes through the centre $\left(\frac{\lambda+4}{4},\frac{-(\lambda+1)}{2}\right)$ of $S_2=0$

$$(\lambda+2)\left(\frac{\lambda+4}{4}\right)+2(\lambda+2)\left(\frac{\lambda+1}{2}\right)-2(\lambda+2)=0$$

$$\lambda(\lambda+2)=0$$

$\lambda=0$ or $\lambda=-2$

when $\lambda=-2$, $S_2 \equiv S_1$

when $\lambda=0$, $S_2 \equiv x^2+y^2-2x+y=0$

(iii) $$x^2 + y^2 - \left(\frac{\lambda_1 + 4}{2}\right)x + (\lambda_1 + 1)y + \lambda_1 = 0$$

$$x^2 + y^2 - \left(\frac{\lambda_2 + 4}{2}\right)x + (\lambda_2 + 1)y + \lambda_2 = 0$$

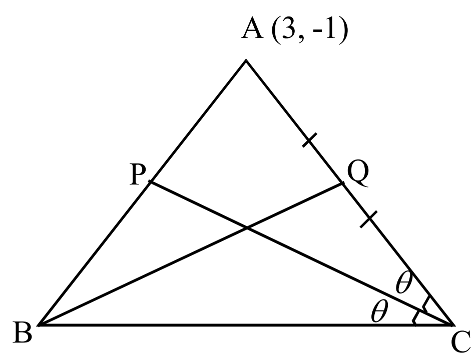
Orthogonal to each other

centre $C_1 \equiv \left(\frac{\lambda_1 + 4}{4}, -\frac{\lambda_1 + 1}{2}\right)$

$C_2 \equiv \left(\frac{\lambda_2 + 4}{4}, -\frac{\lambda_2 + 1}{2}\right)$

$$2\left(\frac{\lambda_1 + 4}{4}\right)\left(\frac{\lambda_2 + 4}{4}\right) + 2\left(\frac{\lambda_1 + 1}{2}\right)\left(\frac{\lambda_2 + 1}{2}\right) = \lambda_1 + \lambda_2$$

$$\lambda_1 \lambda_2 = -4$$



12. (*a*) Equation of CP is $x - 4y + 10 = 0$

Equation of BQ is $6x + 10y - 59 = 0$

C lies on $x - 4y + 10 = 0$

$C \equiv \left(t, \frac{t+10}{4}\right)$, $A \equiv (3, -1)$

$Q \equiv \left(\frac{t+3}{2}, \frac{t+6}{8}\right)$

Since Q lies on BQ, $6x + 10y - 59 = 0$

$$6\left(\frac{t+3}{2}\right) + 10\left(\frac{t+6}{8}\right) - 59 = 0$$

$t = 10$

$C \equiv (10, 5)$

Gradient of AC is $\frac{6}{7}$

Gradient of CP is $\frac{1}{4}$

Let Gradient of BC be m.

$$\left|\frac{m-\frac{1}{4}}{1+\frac{m}{4}}\right| = \left|\frac{\frac{6}{7}-\frac{1}{4}}{1+\frac{6}{7}\times\frac{1}{4}}\right|$$

$$\left|\frac{4m-1}{4+m}\right| = \left|\frac{1}{2}\right|$$

$$\frac{4m-1}{4+m} = \pm\frac{1}{2}$$

$$m = \frac{6}{7} \text{ or } -\frac{2}{9}$$

Equation of BC is $y-5 = -\frac{2}{9}(x-10)$

$$2x+9y-65=0$$

Equation of AC is $y+1 = \frac{6}{7}(x-3)$

$$6x-7y-25=0$$

$$BC: 2x+9y-65=0$$
$$BQ: 6x+10y-59=0$$

$$B \equiv \left(-\frac{7}{2}, 8\right)$$

Equation of the line perpendicular to AC can be written in the form $7x+6y+c=0$

Since this line passes through $B \equiv \left(-\frac{7}{2}, 8\right)$

$$7\times\left(-\frac{7}{2}\right) + 6\times 8 + c = 0$$

$$c = \frac{-47}{2}$$

Equation is $14x+12y-47=0$

(b) Equation of $S_3 = 0$ is

$\left(3x^2+3y^2-6x-1\right)+\lambda\left(x^2+y^2+2x-4y+1\right)=0$

Centre of $S_1 = 0$ is $(1, 0)$

$S_3 = 0$ passes throug $(1, 0)$

$\left(3+0-6-1\right)+\lambda\left(1+0+2-0+1\right)=0$

$\lambda = 1$

$S_3 = 0$

$\left(3x^2+3y^2-6x-1\right)+\lambda\left(x^2+y^2+2x-4y+1\right)=0$

$4x^2+4y^2-4x-4y=0$

$x^2+y^2-x-y=0$

$S_2 = 0 \qquad g = 1 \qquad f = -2 \qquad c = 1$

$S_3 = 0 \qquad g' = -\dfrac{1}{2} \qquad f' = -\dfrac{1}{2} \qquad c' = 0$

$2gg' + 2ff' = 2\times 1\times\left(-\dfrac{1}{2}\right)+2\times(-2)\times\left(-\dfrac{1}{2}\right)=-1+2=1$

$c + c' \quad = \quad 1+0 \quad = \quad 1$

$2gg' + 2ff' = c + c'$

$S_3 = 0$ and $S_2 = 0$ intersect orthogonally.

centre of S_1 is $(1, 0)$

Equation of the tangent at (x_1, y_1) to the

circle $x^2+y^2+2gx+2fy+c=0$

$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0$

Equation of the tangent at $(1, 0)$ to the

circle $x^2+y^2-x-y=0$

$x\times 1+y\times 0-\dfrac{1}{2}(x+1)-\dfrac{1}{2}(y+0)=0$

$x-\dfrac{x+1}{2}-\dfrac{y}{2}=0$

$x-y-1=0$

Equatio of AB is

$$y-8=\left(\frac{-1-8}{3+\frac{7}{2}}\right)\left(\lambda+\frac{7}{2}\right)$$

$$y-8=\frac{9\times 2}{13}\left(\lambda+\frac{7}{2}\right)$$

$$13y-104=-18x-63$$

$$18x+13y-41=0$$

13. (*a*)(i) $(2\sin x-\cos x)(1+\cos x)=\sin^2 x$

$$\left(2\sin x-\cos x\right)\left(1+\cos x\right)-\left(1-\cos^2 x\right)=0$$

$$(1+\cos x)\left[\left(2\sin x-\cos x-\left(1-\cos x\right)\right]=0$$

$$(1+\cos x)\left(2\sin x-1\right)=0$$

$\cos x+1=0$ or $2\sin x-1=0$

$\cos x=-1$ $\qquad \sin x=\frac{1}{2}$

$x=2n\pi\pm\pi, n\in\mathbb{Z}$ $\qquad x=n\pi+\left(-1\right)^n\frac{\pi}{6}; n\in\mathbb{Z}$

(ii) $2\tan x+\sec 2x=2\tan 2x$

$$2\tan x+\frac{1+\tan^2 x}{1-\tan^2 x}=\frac{4\tan x}{1-\tan^2 x}$$

$$2\tan x\left(1-\tan^2 x\right)+\left(1+\tan^2 x\right)=4\tan x$$

$$2\tan x-2\tan^3 x+1+\tan^2 x=4\tan x$$

$$2\tan^3 x-\tan^2 x+2\tan x-1=0$$

$$\tan^2 x\left(2\tan x-1\right)+1\left(2\tan x-1\right)=0$$

$$\left(\tan^2 x+1\right)\left(2\tan x-1\right)=0$$

$\tan^2 x+1\neq 0$; $\quad \tan x=\frac{1}{2}$

$$x=n\pi+\alpha\left[\alpha=\tan^{-1}\left(\frac{1}{2}\right)\right] \quad n\in Z$$

(*b*) $2\cos^2\theta - 2\cos^2 2\theta$

$= (1+\cos 2\theta) - (1+\cos 4\theta)$

$= \cos 2\theta - \cos 4\theta$

$\theta = 36^0$

$2\cos^2 36^0 - 2\cos^2 72^0 = \cos 72^0 - \cos 144^0$

$2\left(\cos 36^0 - \cos 72^0\right)\left(\cos 36 + \cos 72\right) = \cos 72^0 - \cos 144^0$

$$\cos 36^0 - \cos 72^0 = \frac{\cos 72^0 - \cos 144^0}{2\left(\cos 36^0 + \cos 72^0\right)}$$

$$= \frac{2\sin 108^0 \sin 36^0}{4\cos 54^0 \cos 18^0}$$

$$= \frac{2\cos 18^0 \cos 54^0}{4\cos 54^0 \cos 18^0} = \frac{1}{2}$$

$\cos 36 - \cos 72 = \frac{1}{2}$

$\cos 36 = x$

$x - \left(2x^2 - 1\right) = \frac{1}{2}$

$2x - 4x^2 + 2 = 1$

$4x^2 - 2x - 1 = 0$

$$x = \frac{2 \pm \sqrt{4+16}}{8}$$

$$= \frac{2 \pm 2\sqrt{5}}{8}$$

$$= \frac{1 \pm \sqrt{5}}{4}$$

Since $\cos 36^0 > 0$

$$\cos 36^0 = \frac{\sqrt{5}+1}{4}$$

$$\cos 72 = \cos 36 - \frac{1}{2}$$

$$= \frac{\sqrt{5}+1}{4} - \frac{1}{2} = \frac{\sqrt{5}-1}{4}$$

(*c*) (i) $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = k$ (say).

$$\frac{a^2 - b^2}{\cos A + \cos B} + \frac{b^2 - c^2}{\cos B + \cos C} + \frac{c^2 - a^2}{\cos C + \cos A}$$

$$= \frac{k^2\left(\sin^2 A - \sin^2 B\right)}{\cos A + \cos B} + \frac{k^2\left(\sin^2 B - \sin^2 C\right)}{\cos B + \cos C} + \frac{k^2\left(ain^2 C - \sin^2 A\right)}{\cos C + \cos A}$$

$$= \frac{k^2\left(\cos^2 B - \cos^2 A\right)}{\cos A + \cos B} + \frac{k^2\left(\cos^2 C - \cos^2 B\right)}{\cos B + \cos C} + \frac{k^2\left(\cos^2 A - \cos^2 C\right)}{\cos C + \cos A}$$

$$= k^2\left(\cos B - \cos A\right) + k^2\left(\cos C - \cos B\right) + k^2\left(\cos A - \cos C\right) = 0$$

(ii) $\dfrac{a}{\sin 45^0} = \dfrac{b}{\sin 75^0} = \dfrac{c}{\sin 60^0} = t$ (say)

$$a + \sqrt{2}c - 2b$$

$$= t\sin 45^0 + \sqrt{2}t\sin 60^0 - 2t\sin 75^0$$

$$= t\left[\sin 45^0 + \sqrt{2}\sin 60^0 - 2\sin 75^0\right]$$

$$= t\left[\frac{1}{\sqrt{2}} + \sqrt{2}\frac{\sqrt{3}}{2} - 2\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)\right]$$

$$= t\left[\frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{\sqrt{2}} - \frac{\sqrt{3}+1}{\sqrt{2}}\right] = 0$$

$$a + \sqrt{2}c - 2b = 0$$

$$a + \sqrt{2}c = 2b$$

14. (*a*) (i) $2(\cos x + \cos 2x) + \sin 2x(1 + 2\cos x) = 2\sin x$

$$2(\cos x + \cos 2x) + 2\sin x\cos x(1 + 2\cos x) - 2\sin x = 0$$

$$2(\cos x + \cos 2x) + 2\sin x(\cos x + 2\cos^2 x - 1) = 0$$

$$(\cos x + \cos 2x) + \sin x(\cos x + \cos 2x) = 0$$

$$(1 + \sin x)(\cos x + \cos 2x) = 0$$

$\sin x + 1 = 0$ $\qquad$ $\cos x + \cos 2x = 0$

$\sin x = -1$ $\qquad$ $2\cos\frac{3x}{2}.\cos\frac{x}{2} = 0$

$x = -\frac{\pi}{2}$ $\qquad$ $\cos\frac{x}{2} = 0$ or $\cos\frac{3x}{2} = 0$

$\frac{x}{2} = 2n\pi \pm \frac{\pi}{2}$ $\qquad$ $\frac{3x}{2} = 2n\pi \pm \frac{\pi}{2}$

$x = 4n\pi \pm \pi$ $\qquad$ $x = \frac{\pi}{3}(4n \pm 1)$

$x = \pm\pi$ $\qquad$ $x = \pm\frac{\pi}{3}, -\pi$

Solutions: $\pm\frac{\pi}{3}, -\frac{\pi}{3}, \pi$ $\qquad$ $[-\pi < x \leq \pi]$

(ii) $\tan^{-1}\left(\frac{1}{x-1}\right) - \tan^{-1}\left(\frac{1}{x+1}\right) = \tan^{-1}\left(\frac{3}{5}\right) - \tan^{-1}\left(\frac{1}{3}\right)$

$\tan^{-1}\left(\frac{1}{x-1}\right) = A$, $\qquad$ $\tan^{-1}\left(\frac{1}{x+1}\right) = B$

$\tan^{-1}\left(\frac{3}{5}\right) = C$, $\qquad$ $\tan^{-1}\left(\frac{1}{3}\right) = D$

$A\text{-}B = C\text{-}D$

$\tan(A - B) = \tan(C - D)$

$$\frac{\tan A - \tan B}{1 + \tan A \tan B} = \frac{\tan C - \tan D}{1 + \tan C \tan D}$$

$$\frac{\frac{1}{x-1} - \frac{1}{x+1}}{1 + \frac{1}{(x-1)(x+1)}} = \frac{\frac{3}{5} - \frac{1}{3}}{1 + \frac{3}{5} \times \frac{1}{3}}$$

$\frac{2}{x^2} = \frac{4}{18}$

$x^2 = 9$

$x = \pm 3$

Since $2 < x < 4$, $\quad x = 3$

(b) $$\frac{\sin(\theta+\alpha)}{(1-m)}=\frac{\cos(\theta-\alpha)}{(1+m)}$$

$$\frac{\sin(\theta+\alpha)+\cos(\theta-\alpha)}{2}=\frac{\cos(\theta-\alpha)-\sin(\theta+\alpha)}{2m}$$

$$m\left[\sin(\theta+\alpha)+\sin\left[\frac{\pi}{2}-(\theta-\alpha)\right]\right]=\left[\sin\left[\frac{\pi}{2}-(\theta-\alpha)\right]-\sin(\theta+\alpha)\right]$$

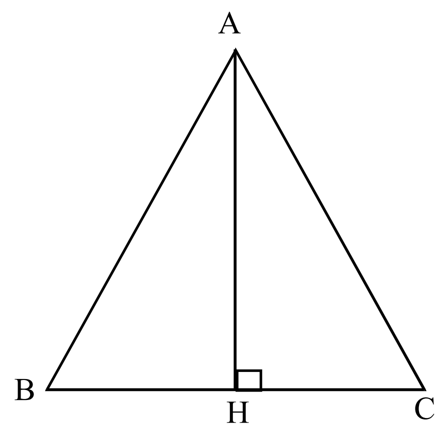
$$m\times 2\sin\left(\frac{\pi}{4}+\alpha\right)\cos\left(\frac{\pi}{4}-\theta\right)=2\cos\left(\frac{\pi}{4}+\alpha\right)\sin\left(\frac{\pi}{4}-\theta\right)$$

$$m\tan\left(\frac{\pi}{4}+\alpha\right)=\tan\left(\frac{\pi}{4}-\theta\right)$$

$$\tan\left(\frac{\pi}{4}-\theta\right)=m.\cot\left[\frac{\pi}{2}-\left(\frac{\pi}{4}+\alpha\right)\right]$$

$$\tan\left(\frac{\pi}{4}-\theta\right)=m.\cot\left(\frac{\pi}{4}-\alpha\right)$$

(c) $$\cos A=\frac{b^2+c^2-a^2}{2bc}$$



(i) $(b+c)^2-a^2$

$$=\left(b^2+2bc+c^2\right)-\left(b^2+c^2-2bc\cos A\right)$$

$$=2bc(1+\cos A)$$

$$=4bc.\cos^2\frac{A}{2} \qquad \text{——————} \qquad (1)$$

Area of the triangle ABC $=\frac{1}{2}bc.\sin A=\frac{1}{2}a.p$

$$bc.\sin A=a.p \qquad \text{——————} \qquad (2)$$

From (1) and (2)

$$\left(b+c\right)^2 - a^2 = 4bc.\cos^2\frac{A}{2}$$

$$= \frac{4ap}{\sin A}.\cos^2\frac{A}{2}$$

$$= \frac{4ap}{2\sin\frac{A}{2}\cos\frac{A}{2}}.\cos^2\frac{A}{2}$$

$$= 2ap\cot\frac{A}{2}$$

$$\left(b+c\right)^2 = a^2 + 2ap.\cot\frac{A}{2}$$

(ii) $a^4 + b^4 + c^4 - 2b^2c^2 - 2c^2a^2 + 2a^2b^2 = 2a^2b^2$

$$\left(a^2+b^2-c^2\right)^2 = 2a^2b^2$$

$$a^2+b^2-c^2 = \pm ab\sqrt{2}$$

$$\frac{a^2+b^2-c^2}{2ab} = \frac{\pm ab\sqrt{2}}{2ab} = \pm\frac{1}{\sqrt{2}}$$

$$\cos c = \pm\frac{1}{\sqrt{2}}, \qquad c = \frac{\pi}{4} \text{ or } \frac{3\pi}{4}$$

15. (a) $A = \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}$

$$A^2 = \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 8 & 5 \\ -5 & 3 \end{pmatrix}$$

Now $A^2 - 5A + 7I$

$$A^2 = \begin{pmatrix} 8 & 5 \\ -5 & 3 \end{pmatrix} - \begin{pmatrix} 15 & 5 \\ -5 & 10 \end{pmatrix} + \begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$A^2 - 5A + 7I = 0$$

$$7I = 5A - A^2 \qquad\qquad 7I = 5A - A^2$$

$$= A(5I - A) \qquad\qquad = (5I - A)A$$

$$I = A.\frac{1}{7}(5I - A) \qquad\qquad I = \frac{1}{7}(5I - A)A$$

Hence $A^{-1} = \frac{1}{7}(5I - A)$

$$= \frac{1}{7}\left[\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} - \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}\right]$$

$$= \frac{1}{7}\begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} \frac{2}{7} & \frac{-1}{7} \\ \frac{1}{7} & \frac{3}{7} \end{pmatrix}$$

$BA = C$

$(BA)A^{-1} = CA^{-1}$

$B(AA^{-1}) = CA^{-1}$

$$B = CA^{-1} = \begin{pmatrix} 9 & -4 \\ 6 & 16 \end{pmatrix}\begin{pmatrix} \frac{2}{7} & \frac{-1}{7} \\ \frac{1}{7} & \frac{3}{7} \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ 4 & 6 \end{pmatrix}$$

(b) $x - y = a$ .... (1)

$x + y = b$ .... (2)

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$A \quad X \quad = \quad B$$

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \end{pmatrix}, \quad B = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$A^{-1} = \frac{1}{[1-(-1)]}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$AX = B$$

$$A^1 AX = A^{-1}B$$

$$X = A^{-1}B$$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\therefore x = \frac{a}{2} + \frac{b}{2}$$

$$y = -\frac{a}{2} + \frac{b}{2}$$

$$A^2 X = B$$

$$A^1 A^2 X = A^{-1}B$$

$$\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} \frac{b}{2} \\ -\frac{a}{2} \end{pmatrix}$$

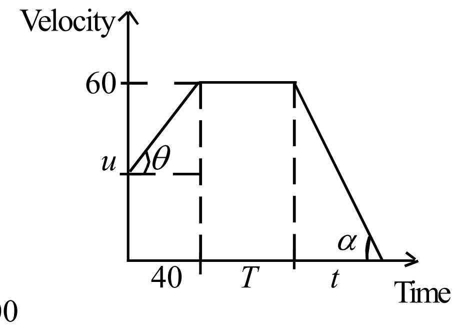
$$p = \frac{b}{2}$$

$$q = -\frac{a}{2}$$

Combained mathematics II

Part A

01. $\tan\theta = 1 = \dfrac{60-u}{40}$



$u = 20ms^{-1}$

$\tan\alpha = \dfrac{1}{2} = \dfrac{60}{t}$

$t = 120$

Distance bavelled *10000 m*

$\dfrac{1}{2}[60+u]\times 40 + 60\times T + \dfrac{1}{2}\times 60\times t = 10000$

$80\times 20 + 60T + 30\times 120 = 10000$

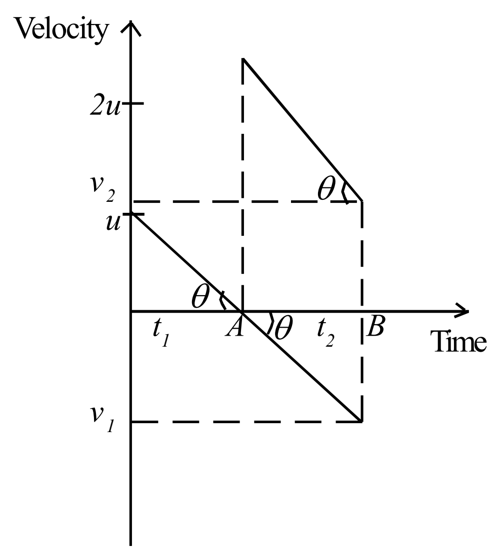
$T = 80$ Seconds

02. $OA = t_1, \quad AB = t_2$

$\tan\theta = g = \dfrac{u}{t_1}, \quad t_1 = \dfrac{u}{g}$

$\tan\theta = g = \dfrac{v_1}{t_2}, \quad t_2 = \dfrac{v_1}{g}$

$\tan\theta = g = \dfrac{2u - v_2}{t_2}, \quad v_2 = 2u - gt_2$



Displacement of $A = \dfrac{1}{2}ut_1 - \dfrac{1}{2}v_1t_2$

Displacement of $B = \dfrac{1}{2}(2u + v_2)t_2$

$\dfrac{1}{2}u.t_1 - \dfrac{1}{2}v_1.t_2 = \dfrac{1}{2}(2u+v_2)t_2$

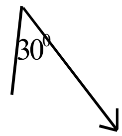
$u.t_1 - v_1.t_2 = (2u+v_2)t_2$

$u.t_1 = (2u + v_1 + v_2)t_2$

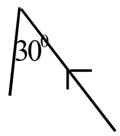
$u.\dfrac{u}{g} = (2u + gt_2 + 2u - gt_2)t_2$

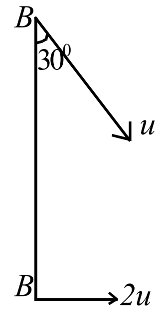
$t_2 = \dfrac{u}{4g}$

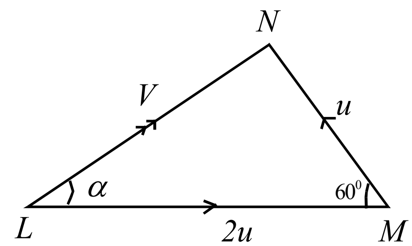
03. $V_{A,E} = \rightarrow 2u$

$V_{B,E} =$ 

$V_{A,B} = V_{A,E} + V_{E,B}$

$V = \xrightarrow[2u]{} +$ 





$$v^2 = LN^2 = (2u)^2 + (u)^2 - 2\times 2u\times u.\cos 60$$

$$= 3u^2$$

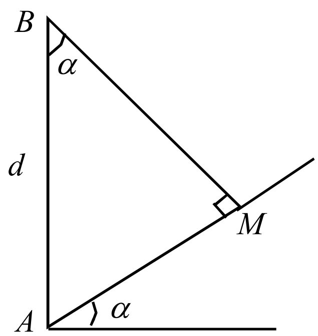
$$V = \sqrt{3}u$$

$$\frac{u}{\sin\alpha} = \frac{v}{\sin 60}$$

$$\frac{u}{\sin\alpha} = \frac{\sqrt{3}u\times 2}{\sqrt{3}}$$

$$\sin\alpha = \frac{1}{2}$$

$$\alpha = 30^0$$



Shortest distance $= d\cos\alpha = d\cos 30 = \dfrac{\sqrt{3}d}{2}$

Time taken $= \dfrac{d\sin 30}{v} = \dfrac{d}{2\sqrt{3}u}$

04. $x + 2y =$ constant

$\dot{x} + 2\dot{y} = 0$

$\ddot{x} + 2\ddot{y} = 0$

Let $\ddot{y} = a$. then $\ddot{x} = -2a$

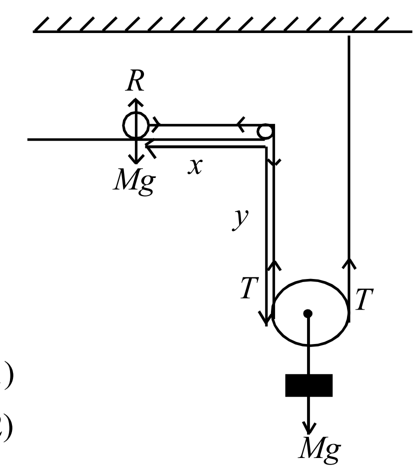
i.e acceleration $AM, E = \downarrow a$

$Am, E = \rightarrow 2a$

Applying $F = ma$

$M \downarrow, \quad Mg - 2T = Ma$ ———————— (1)

$\xrightarrow[m]{}, \quad T = m(2a)$ ———————— (2)



$$a = \frac{M_g}{4M+m}, \quad T = \frac{2Mmg}{M+4m}$$

05. $\underline{r} = a\cos nt\,\underline{i} + b\sin nt\,\underline{j}$

$$\frac{d\underline{r}}{dt} = \underline{v} = -an\sin nt\,\underline{i} + bn\cos nt\,\underline{j}$$

$$\frac{dv}{dt} = \underline{f} = -an^2\cos nt\,\underline{i} - bn^2\sin nt\,\underline{j} = -n^2\left[a\cos nt\,\underline{i} + b\sin nt\,\underline{j}\right]$$

If $\underline{v}$ is perpendicular to $\underline{f}$ then $\underline{v}.\underline{f} = 0$

$$a^2n^3\cos nt\sin nt - b^2n^3\sin nt\cos nt = 0$$

$$\frac{1}{2}\left(b^2 - a^2\right)n^3\sin 2nt = 0$$

$$t = \frac{k\pi}{2n};\ \text{where } k = 0,1,2,3...$$

$$\underline{V}..\underline{V} = a^2n^2\sin^2 nt + b^2n^2\cos^2 nt$$

$$= n^2\left[a^2\sin^2 nt + b^2\cos^2 nt\right]$$

$$\underline{r}.\underline{r} = a^2\cos^2 nt + b^2\sin^2 nt$$

$$a^2 + b^2 - \underline{r}.\underline{r} = a^2\sin^2 nt + b^2\cos^2 nt$$

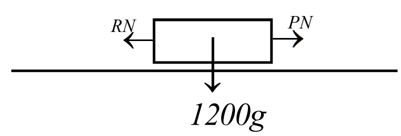
$$\underline{V}..\underline{V} = n^2\left(a^2 + b^2 - \underline{r}.\underline{r}\right)$$

06. Constant velocity means that acceleration is zero,

Applying $F = ma$

$\rightarrow P - 600 = 1200\times 0$

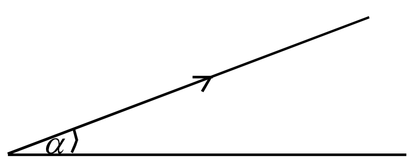
$P = 600N$

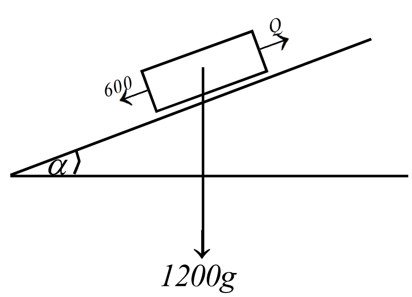


Power $= 600\times\frac{20}{3} = 4000$ Watts.

$= 4kW$

Applying $F = ma$





$$Q - 600 - 1200\times 10\sin\alpha = 1200a$$

$$Q = 600 + 1200\times 10\times\frac{1}{24} + 1200a$$

$$= \left(1200a + 1100\right)$$

$Q \times 20 = 30 \times 1000$

$(1200a + 1100) \times 20 = 30 \times 1000$

$$a = \frac{1}{3} ms^{-2}$$

07. Let the velocity of water be Vms^{-1}

$\frac{100}{100 \times 100} \times V = \frac{1}{10}$

$V = 10ms^{-1}$

Power $=$ Workdone in one second

$=$ Change in energy in one second

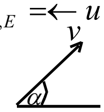
$= \frac{1}{2}mv^2 + mgh$

$= \frac{1}{2}(0.1 \times 1000) \times 10^2 + (0.1 \times 1000) \times 10 \times 12$

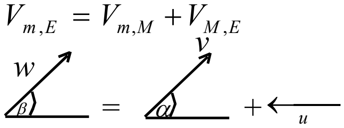
$= 17000$ Watts

$= 17kW$

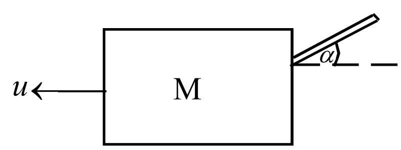
08. Let $V_{M,E} = \leftarrow u$

$V_{m,M} =$ 

$V_{m,E} = V_{m,M} + V_{M,E}$



Have of conservation of Momentam



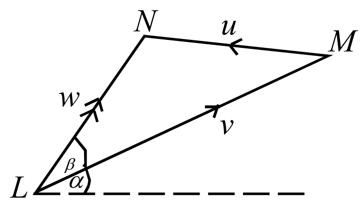
(for m and M)

(M, m) $\leftarrow$

$Mu - m(v\cos\alpha - u) = 0$

$u = \frac{mv\cos\alpha}{M + m}$

Velocity triangle from
sine rule



$$\frac{v}{\sin(180-\beta)} = \frac{u}{\sin(\beta-\alpha)}$$

$$\frac{v}{\sin\beta} = \frac{mv\cos\alpha}{(M+m)\sin(\beta-\alpha)}$$

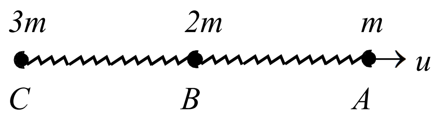
$$(M+m)\sin(\beta-\alpha) = m\sin\beta\cos\alpha$$

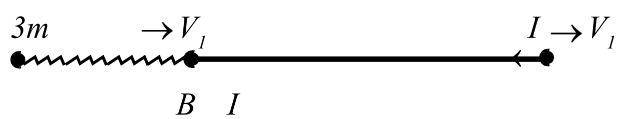
$$(M+m)\left[\sin\beta\cos\alpha - \cos\beta\sin\alpha\right] = m\sin\beta\cos\alpha$$

$$M\sin\beta\cos\alpha = (M+m)\cos\beta\sin\alpha$$

$$\tan\beta = \frac{M+m}{M}\tan\alpha$$

09.

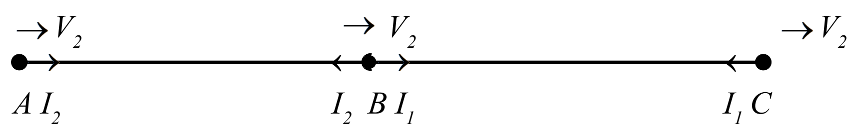




Using $I = \Delta mv$ for the system

$\rightarrow m(v_1 - u) + 2m(v_1 - 0) = 0 \qquad \therefore mv_1 + 2mv_1 = mu$

$$v_1 = \frac{u}{3}$$



Using $I = \Delta mv$ for the system

$$0 = m(v_2 - v_1) + 2m(v_2 - v_1) + 3m(v_2 - 0)$$

$$v_2 = \frac{u}{6}$$

Applying $\underline{I} = \Delta(m\underline{v})$

for A , $\qquad \rightarrow -I_1 = m(V_2 - V_1)$

$$= m\left(\frac{u}{6} - \frac{u}{3}\right)$$

$$I_1 = \frac{mu}{6}$$

Applying C, $\rightarrow + I_2 = 3m(V_2 - V_0)$

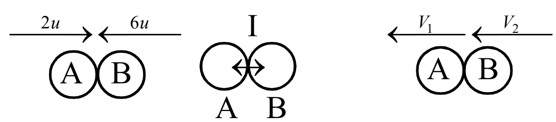
$$= m\left(\frac{u}{6} - 0\right)$$

$$I_1 = \frac{3mu}{6}$$

$$I_2 : I_1 = 3:1$$

Loss of K.E $= \frac{1}{2}mu^2 - \left[\frac{1}{2}m\left(\frac{u}{6}\right)^2 + \frac{1}{2}2m\left(\frac{u}{6}\right)^2 + \frac{1}{2}3m\left(\frac{u}{6}\right)^2\right]$

$$= \frac{5}{12}mu^2$$

10. 

Using $I = \Delta mv$ for the system

$\leftarrow m(v_1 - 2u) + 4m(v_2 - 6u) = 0 \qquad \therefore mv_1 + 4mv_2 = 4m \times 6u - m \times 2u$

$v_1 + 4v_2 = 22u$ ——————— (1)

Newton's Law of restitution

$$v_1 - v_2 = \frac{1}{2}(6u + 2u)$$

$$v_1 - v_2 = 4u$$

$v_2 = \frac{18u}{5}$ ——————— (2)

Applying $\underline{I} = \Delta(m\underline{v})$

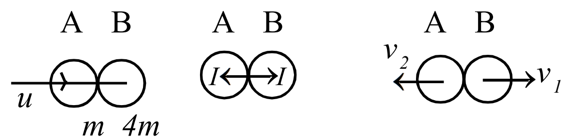
$\xleftarrow{B}$, $\quad -I_1 = 4m(v_2 - 6u)$

$$= 4m\left(\frac{18u}{5} - 6u\right)$$

$$I = \frac{48mu}{5}$$

Hence, momentum transferred is $\frac{48mu}{5}$

11.



Using $I = \Delta mv$ for the system

$\rightarrow m(v_2 - u) + 4m(v_1 - 0) = 0 \qquad \therefore 4mv_1 - mv_2 = mu$

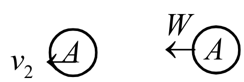
$4v_1 - v_2 = u$ ——— (1)

$v_1 + v_2 = eu$ ——— (2)

$$v_1 = \frac{(1+e)u}{5}, \quad v_2 = \frac{(4e-1)u}{5}$$

$v_2 > 0$ implies that $e > \frac{1}{4}$ ——— (3)

Let velocity of B after the impact with the wall be W.



$$W = ev_1 = \frac{4}{5}\left(\frac{1+e}{5}\right)u$$

For the second collision $W > V_2$

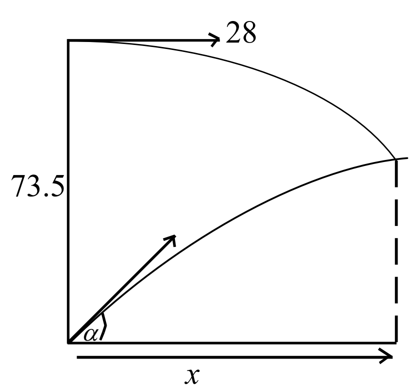
$$\frac{4}{5} = \left(\frac{1+e}{5}\right)u > \frac{4e-1}{5}u$$

$$4(1+e) > 5(4e-1)$$

$e < \frac{9}{16}$ ——— (4)

From (3) and (4) $\frac{1}{4} < e < \frac{9}{16}$

12. Let t be the time taken for the collision



$$S = ut + \frac{1}{2}at^2$$

For A,

$\longrightarrow \qquad x = 28t$

For B,

$\longrightarrow \qquad x = 35\cos\alpha t$

$$35\cos\alpha = 28$$

$$\cos\alpha = \frac{4}{5} \qquad\text{———————} (1)$$

For A, $\quad h_1 = 0 + \frac{1}{2}gt^2$

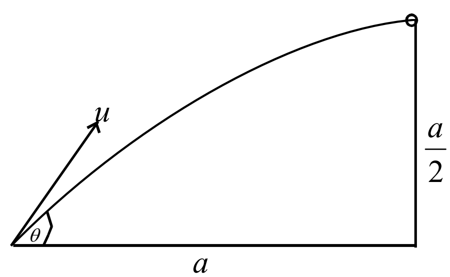
For B, $\quad h_2 = 35\sin\alpha t - \frac{1}{2}gt^2$

$\downarrow \quad h_1 + h_2 = 35\sin\alpha.t$

$\uparrow \quad 73.5 = 35 \times \frac{3}{5} \times t$

$t = 3.5$ seconds

13. $S = ut + \frac{1}{2}at^2$



$u = \sqrt{2ag}$

$\longrightarrow$

$a = u\cos\theta t \qquad\text{———————} (1)$

$\uparrow \quad \frac{a}{2} = u\sin\theta t - \frac{1}{2}gt^2$

From (1), $\quad t = \frac{a}{u\cos\theta}$

$$\frac{a}{2} = a\tan\theta - \frac{ga^2}{2u^2\cos^2\theta}$$

$$\frac{a}{2} = a\tan\theta - \frac{a}{4}\left(1 + \tan^2\theta\right)$$

$$\tan^2\theta - 4\tan\theta + 3 = 0$$

$$(\tan\theta - 3)(\tan\theta - 1) = 0$$

$\tan\theta = 3$ or $\tan\theta = 1$

$$S = ut + \frac{1}{2}at^2$$

For A, $\longrightarrow \quad a = u\cos\theta_1.t_1$

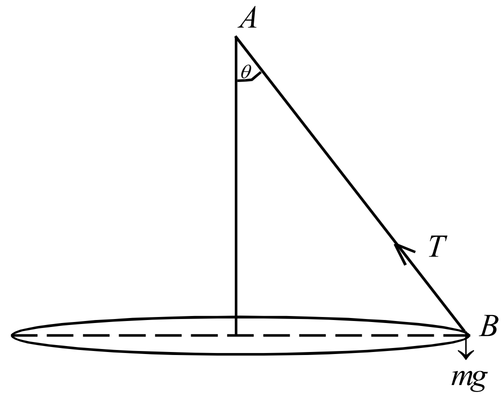
For B, $\longrightarrow \quad a = u\cos\theta_2.t_2$

$$\frac{t_1}{t_2} = \frac{\cos\theta_2}{\cos\theta_1} = \frac{1}{\sqrt{10}} \times \frac{\sqrt{2}}{1} = \frac{1}{\sqrt{5}}$$

$$t_1 : t_2 = 1 : \sqrt{5}$$

14. Let $AB = l$

Natural length $= a$, $\lambda = 2mg$



$$T = \frac{2mg(l-a)}{a}$$

Applying $F = ma$

$\overset{B}{\longleftarrow}$, $T\sin\theta = ml\sin\theta.\left(\frac{3g}{4a}\right)$

$$T = \frac{3mgl}{4a} \text{ ——————— } (1)$$

From (1) and (2)

$$\frac{2mg(l-a)}{a} = \frac{3mgl}{4a}$$

$$l = \frac{8a}{5} \text{ ——————— } (2)$$

Extension is $\frac{8a}{5} - a = \frac{3a}{5}$

$\uparrow$ $F = ma$

$$T\cos\theta - mg = m \times 0$$

$$T\cos\theta = mg$$

$$\frac{6mg}{5}\cos\theta = mg$$

$$\cos\theta = \frac{5}{6}$$

15. Energy equation

Energy at A = Energy at B

$$\left(\frac{1}{2}mv^2 + mgh = \text{constant}\right)$$

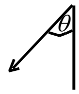
$$O + mga = \frac{1}{2}mw^2 - mga$$

$$w^2 = 4ag = 4 \times 10 \times 0.6$$

$$w^2 = 24$$

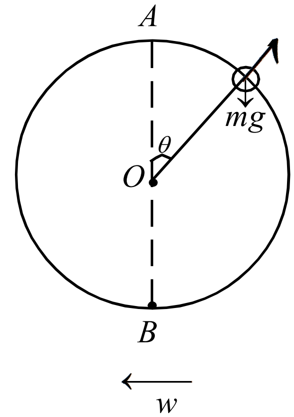
$$w = 2\sqrt{6}ms^{-1}$$

Applying $F = ma$



$$mg\cos\theta - R = \frac{mv^2}{a}$$

$$R = mg\cos\theta - \frac{mv^2}{a} \quad \text{——— (1)}$$



Energy equation

$$O + mga = \frac{1}{2}mv^2 + mga\cos\theta$$

$$v^2 = 2ag(1-\cos\theta) \quad \text{——— (2)}$$

From (1) and (2)

$$R = mg\left(3\cos\theta - 2\right)$$

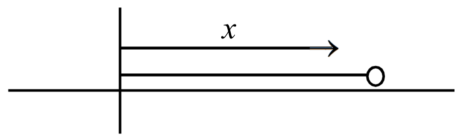
When $R = 0$, $\cos\theta = \frac{2}{3}$

Height is $= 0.6\cos\theta = 0.6\times\frac{2}{3}$

$= 0.4m$

16. $\ddot{x} = -\omega^2 x$

$v^2 = \omega^2\left(a^2 - x^2\right)$



When $x = 0.9$, $v = 1.2$

$x = 1.2$, $v = 0.9$

$$1.2^2 = \omega^2\left(a^2 - 0.9^2\right) \quad \text{——— (1)}$$

$$0.9^2 = \omega^2\left(a^2 - 1.2^2\right) \quad \text{——— (2)}$$

(1) ÷ (2), $\frac{1.2^2}{0.9^2} = \frac{a^2 - 0.9^2}{a^2 - 1.2^2}$

$$a^2\left(1.2^2 - 0.9^2\right) = 1.2^4 - 0.9^4$$

$$a^2 = 1.2^2 + 0.9^2$$

$$a = 1.5m$$

Amplitude $= 1.5m$

$$\omega^2\left(1.5^2 - 0.9^2\right) = 1.2^2$$

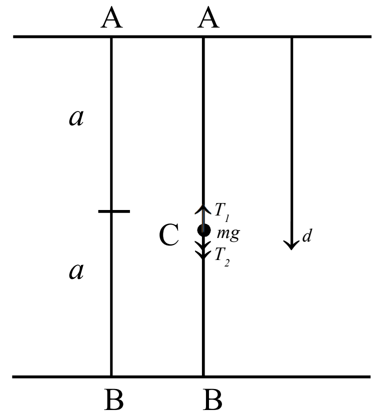
$$\omega^2 = 1$$

$$\omega = 1$$

period $T = \frac{2\pi}{\omega} = 2\pi$ Seconds

17. In equilibrium $AC = d,\ \ \lambda = mg$

For equilibrium of the particle,



$\downarrow,\ T_2 + mg - T_1 = 0$

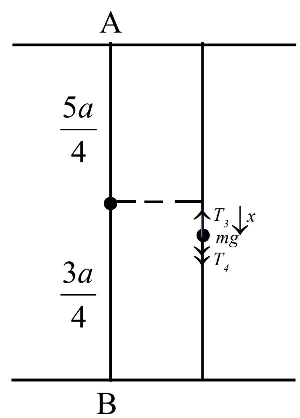
$$\frac{2mg}{a}\left(2a - d - \frac{a}{2}\right) + mg - \frac{2mg}{a}\left(d - \frac{a}{2}\right) = 0$$

$$\frac{2}{a}\left(2a - d - \frac{a}{2}\right) + 1 - \frac{2}{a}\left(d - \frac{a}{2}\right) = 0$$

$$\frac{2}{a}\left(2a - d - \frac{a}{2} - d + \frac{a}{2}\right) + 1 = 0$$

$$d = \frac{5a}{4}, AM = \frac{5a}{4}, BM = \frac{3a}{4}$$

Applying $F = ma$



$\downarrow\ T_4 + mg - T_3 = m\ddot{x}$

$$\frac{2mg}{a}\left[\frac{3a}{4} - x - \frac{a}{2}\right] + mg - \frac{2mg}{a}\left[\frac{5a}{4} + x - \frac{a}{2}\right]$$

$$= m\ddot{x}$$

$$\frac{2g}{a}\left[\frac{3a}{4} - x - \frac{a}{2}\right] + g - \frac{2g}{a}\left[\frac{5a}{4} + x - \frac{a}{2}\right] = \ddot{x}$$

$$\frac{2g}{a}\left[\frac{3a}{4} - x - \frac{a}{2} - \frac{5a}{4} - x + \frac{a}{2}\right] + g = \ddot{x}$$

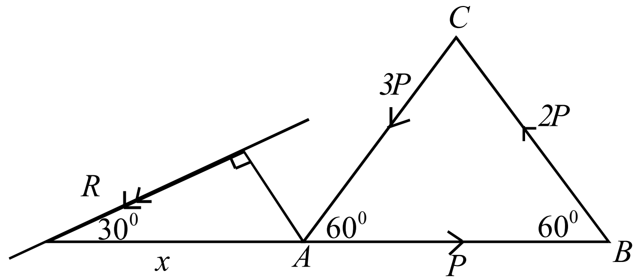
$$\frac{2g}{a}\left[-2x - \frac{a}{2}\right] + g = \ddot{x}$$

$$\ddot{x} = -\frac{4g}{a}x$$

$$\ddot{x} = -\omega^2 x \qquad \left[\omega^2 = \frac{4g}{a}\right]$$

$$\text{Time } = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{a}{4g}} = \pi\sqrt{\frac{a}{g}}$$

18.



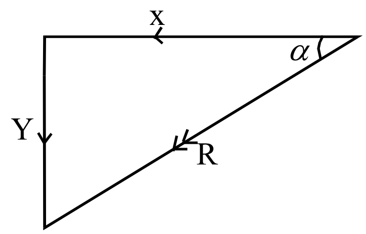
$\longleftarrow$ $X = 3P\cos 60 + 2P\cos 60 - P = \dfrac{3P}{2}$

$\downarrow$ $Y = 3P\sin 60 - 2P\sin 60 = \dfrac{\sqrt{3}P}{2}$

$R^2 = \left(\dfrac{3P}{2}\right)^2 + \left(\dfrac{\sqrt{3}P}{2}\right)^2 = 3P^2$

$R = \sqrt{3}P.N$

$\tan\alpha = \dfrac{Y}{X} = \dfrac{1}{\sqrt{3}}, \ \alpha = 30^0$



Taking moments about A,
Moment of the resultant about A = Moment of the system about A

$\circlearrowleft$ $R.x.\sin 30 = 2P.2a\sin 60$

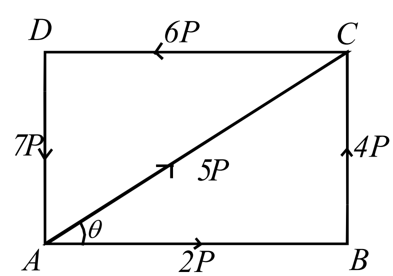
$P\sqrt{3} \times x \times \dfrac{1}{2} = 2P \times 2a \times \dfrac{\sqrt{3}}{2}$

$x = 4a$

19. $\cos\theta = \dfrac{4}{5}, \quad \sin\theta = \dfrac{3}{5}$

$\longrightarrow$ $X = 2P - 6P + 5P\cos\theta$
$= 2P - 6P + 4P = 0$

$\uparrow$ $Y = 4P - 7P + 3P = 0$
$= 4P - 7P + 3P = 0$



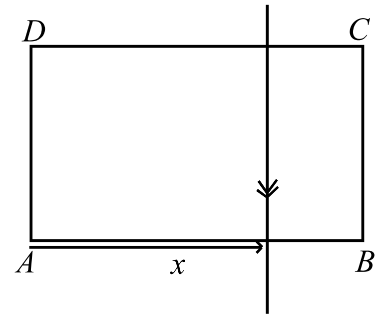
Moment about A,

$G\curvearrowleft = 4P \times 4a + 3P \times 3a = 34Pa$

$R = 0, \quad G \neq 0$

Hence the system reduces to a couple of moment $= 34Pa\curvearrowleft$

If $4P$ is removed, the resultant will be 4P in the direction of CB, *Parallel to* CB



Moment of the resultant about A

= Moment of the system about A

$-4P.x = 18Pa$

$x = -\dfrac{9a}{2}$

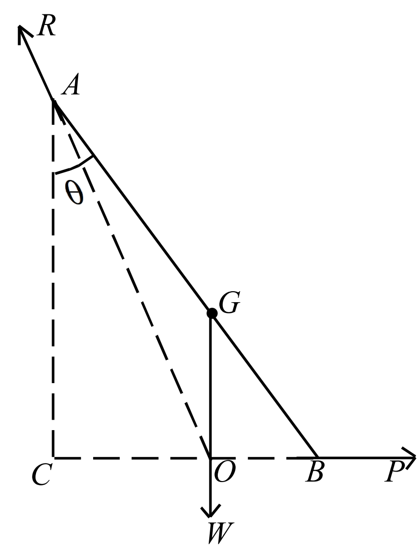
The resultant meets BA lproduced at a distance $\dfrac{9a}{2}$ from A.

20. The forces acting on the rod,

(i) weight W

(ii) Horizontal force P

(iii) Reaction R at R



Consider the triangle OAC

$R \longrightarrow OA$ (R is represented by OA)

$W \longrightarrow AC$ (W is represented by AC)

$P \longrightarrow CO$ (P is represented by CO)

$\tan\theta = \dfrac{3}{4}$

$\dfrac{R}{OA} = \dfrac{W}{AC} = \dfrac{P}{CO}$

If $AB = 2a$

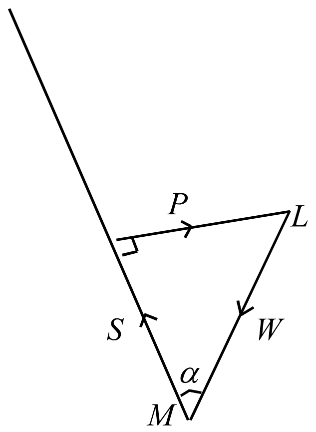
$AC = 2a\sin\theta = \dfrac{8a}{5}$

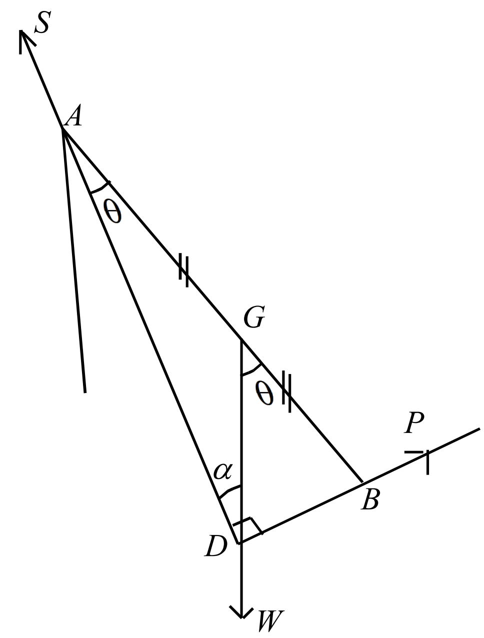
$CB = 2a\sin\theta = \dfrac{6a}{5}$

$CO = \dfrac{3a}{5}$

$P = W.\dfrac{CO}{AC} = \dfrac{3W}{8}$

For equilibrium of AB, W, P and S meet at a point





Consider the triangle of force.
For least value P, P should be perpendicular to S

In the triangle ADB,

$AG = GB$, and $\angle ADB = 90^0$

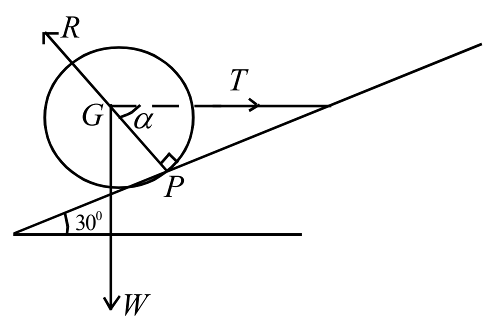
Implies that $AG = GB = GD$

and $\alpha = \dfrac{\theta}{2}$

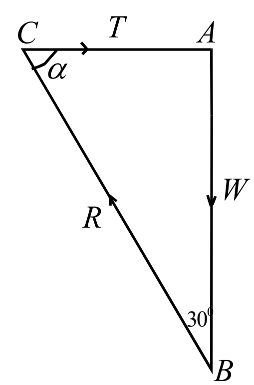
$$P = W \sin\alpha = W \sin\frac{\theta}{2}$$

21. For equilibrium of the sphere,
(i) Weight W at G.
(ii) Reaction R at P
(iii) Tension T

three forces meet at G



$\tan\alpha = \dfrac{12}{9} = \dfrac{4}{3}$ $\sin\alpha = \dfrac{4}{5}$ $\cos\alpha = \dfrac{3}{5}$



Sine rule for the triangle ABC,

$$\frac{T}{\sin 30} = \frac{W}{\sin\alpha} = \frac{R}{\sin(30+\alpha)}$$

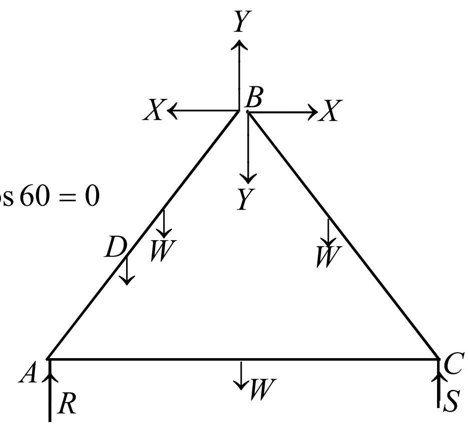
$$T = \frac{W \sin 30}{\sin\alpha} = \frac{5W}{8}$$

$$R = \frac{\sin(30+\alpha)}{\sin\alpha}$$

$$= \frac{W[\sin 30\cos\alpha + \cos 30\sin\alpha]}{\sin\alpha}$$

$$= \frac{W}{8}\left(3+4\sqrt{3}\right)$$

22. For equilibrium of AB,



$A\circlearrowright = 0$

$$X.a\sin 60 + Y.a\cos 60 - W.\frac{a}{2}\cos 60 - W.\frac{a}{3}\cos 60 = 0$$

$$\sqrt{3}\times + Y = \frac{W}{2} + \frac{W}{3} = \frac{5W}{6} \text{ ——————— } (1)$$

For equilibrium of BC

$C\circlearrowright = 0$

$$-X.a\sin 60 + Y.a\cos 60 - W.\frac{a}{2}\cos 60 = 0$$

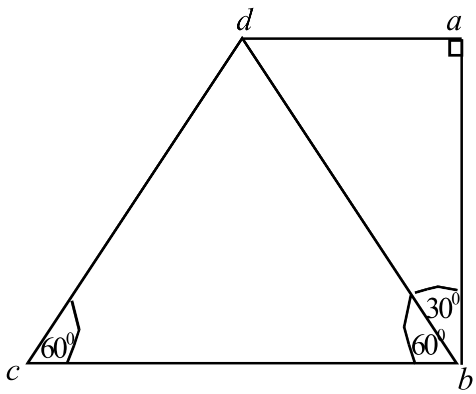
$$-\sqrt{3}X + Y = -\frac{W}{2} \text{ ——————— } (2)$$

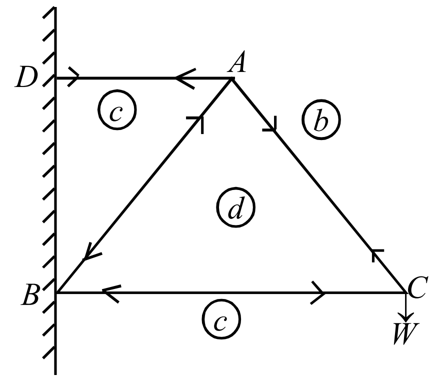
From (1) and (2) $\quad Y = \frac{W}{6} \qquad X = \frac{2W}{3\sqrt{3}}$

Reacton at B is $\sqrt{X^2+Y^2}$

$$= \frac{W\sqrt{57}}{18}$$

23.





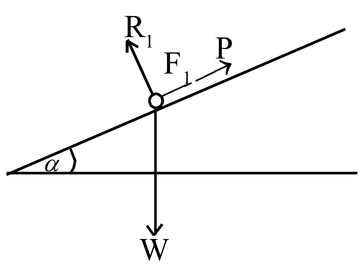
$ab \rightarrow W$

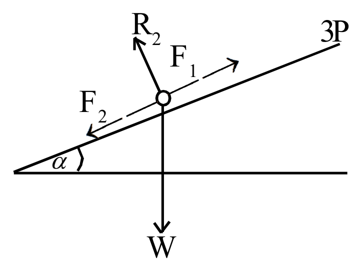
$ad \rightarrow W \tan 30 = \dfrac{W}{\sqrt{3}}$

$bd \rightarrow \dfrac{W}{\cos 30} = \dfrac{2W}{\sqrt{3}}$

$bd = bc = cd$

Rod	Tension	Thrust
BC	–	$\dfrac{W}{\sqrt{3}}$
AC	$\dfrac{2W}{\sqrt{3}}$	-
AB	–	$\dfrac{2W}{\sqrt{3}}$
AD	$\dfrac{2W}{\sqrt{3}}$	-





24. For equilibrium,

$F_1 + P - W \sin\alpha = 0$

$R_1 - W\cos\alpha = 0$

$F_1 = \mu R_1$

$W\sin\alpha - P = \mu W\cos\alpha$

For equilibrium,

$3P - F_2 - W\sin\alpha = 0$

$R_2 - W\cos\alpha = 0$

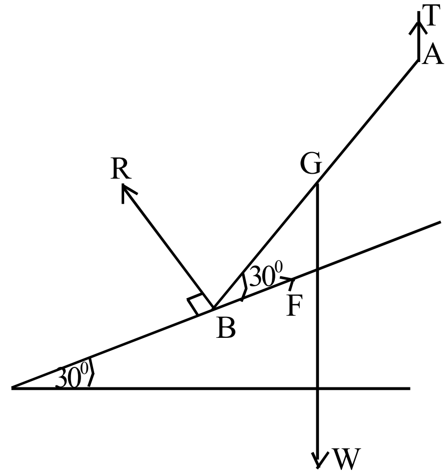
$F_2 = \mu R_2$

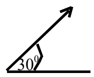
$3P - W\sin\alpha = \mu W\cos\alpha$ ——— (2)

From (1) and (2) $P = \dfrac{W\sin\alpha}{2}$ ——— (1)

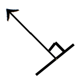
$2\mu = \tan\alpha$

25. For equilibrium of rod AB





$F + T\cos 60 - W\sin 30 = 0$ ——— (1)



$R + T\sin 60 - W\cos 30 = 0$

Moment about B is equal to zero.

$T.2a\cos 60 - Wa\cos 60 = 0$

$T = \dfrac{W}{2}$

$F = W\sin 30 - T\cos 60 = \dfrac{W}{4}$

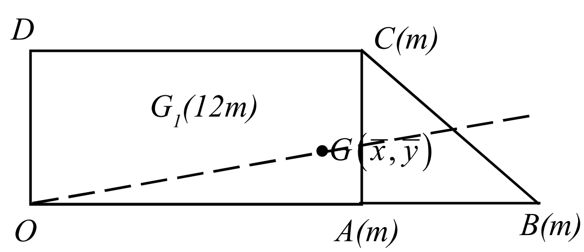
$R = W\cos 30 - T\sin 60 = \dfrac{W\sqrt{3}}{4}$

$\dfrac{F}{R} \le \mu,\quad \mu \ge \dfrac{1}{\sqrt{3}},\ \mu \min\ \dfrac{1}{\sqrt{3}}$

26. Area of rectangle $OACD$ $= 2a^2$

Area of triangle ABC $= \dfrac{1}{2}a^2$

Let the mass of the rectangle $OACD$ is $12m$
Then the mass of the triangle ABC is $3m$
Let $G \equiv \left(\overline{x}, \overline{y}\right)$



Taking moments about OB,

$15m\overline{y} = 12m \times \dfrac{a}{2} + m \times a$

$\overline{y} = \dfrac{7a}{15}$

Taking moments about OD,
$15m\overline{x} = 12m \times a + m \times 2a + m \times 2a + m \times 3a$

$\overline{x} = \dfrac{19a}{15}$

Angle, OA makes with horizontal is β.

$$\tan\beta = \cot\alpha = \frac{\overline{x}}{\overline{y}} = \frac{19a}{7}$$

$$\tan\beta = \tan^{-1}\left(\frac{19}{7}\right)$$

27. $P(B') = \frac{2}{3}, \quad P(AUB) = \frac{5}{8}, \quad P\left(\frac{A}{B}\right) = \frac{3}{4}$

$$P(B) = 1 - P(B') = 1 - \frac{2}{3} = \frac{1}{3}$$

$$P\left(\frac{A}{B}\right) = \frac{P(A\cap B)}{P(B)}$$

$$P(A\cap B) = P\left(\frac{A}{B}\right).P(B) = \frac{3}{4}\times\frac{1}{3} = \frac{1}{4}$$

$$P(A\cup B) = P(A) + P(B) - P(A\cap B); \rightarrow P(A) = \frac{5}{8} - \frac{1}{3} + \frac{1}{4} = \frac{13}{24}$$

$$P(A'\cup B') = P[(A\cap B)'] = 1 - P(A\cap B) = 1 - \frac{1}{4} = \frac{3}{4}$$

28. A and B are independent

$$P(A\cap B) = P(A).P(B) = 0.3\times 0.4 = 0.12$$

$$P(A\cup B) = P(A) + P(B) = P(A\cap B)$$

$$= 0.3 + 0.4 - 0.12$$

$$= 0.58$$

$$P(A'\cap B') = P\left[(A\cup B)'\right] = 1 - P(A\cup B)$$

$$= 1 - 0.58 = 0.42$$

P [one defective] $= \frac{20}{100} = \frac{1}{5}$

P [3defective out of 4] $= 4C_3\left(\frac{1}{5}\right)^3\times\frac{4}{5}$

$$= \frac{16}{625}$$

29. Mean $= \dfrac{7+11+5+8+13+12+11+9+14}{9}$

$\overline{x} = \dfrac{90}{9} = 10$

5 7 8 9 11 11 12 13 14

Median $= \dfrac{9+1}{2}$ th score

= 5 th score = 11

standard deviation $\sigma = \sqrt{\dfrac{\sum_{i=1}^{9}(xi-\overline{x})^2}{n}}$

$$\sigma = \sqrt{\frac{25+9+4+1+1+1+4+9+16}{9}}$$

$$= \frac{\sqrt{70}}{9} = \sqrt{\frac{70}{3}} = 2.78$$

Coefficient of skewness $= \dfrac{3(\text{mean - median})}{\text{standard deviation}}$

$$= \frac{3(10-11)}{2.78}$$

$$= -1.04$$

30.

0	2	(1)
1	1 5 7 9	(4)
2	1 3 8 9	(4)
3	2 3 3 5 6 6 7 9 9 9 9	(11)
4	0 5 7 7 8 9	(6)
5	8	(1)
		27

2/3 means 23 years.

(i) Minimum value 02 yrs.

Maximum value 58 yrs.

Mode 39 yrs.

(ii) Q_1 is the $= \frac{1}{4}(27+1)^{th}$ score

$= 7^{th}$ score $= 23$ yrs.

Median

Q_2 is the $= \frac{1}{2}(27+1)^{th}$ score

$= 14^{th}$ score $= 36$ yrs.

Q_3 is the $= \frac{3}{4}(27+1)$ score

$= 21^{th}$ score $= 40$ yrs.

(iii) $Q_1 - 1.5(Q_3 - Q_1) = 23 + 1.5(40 - 23)$

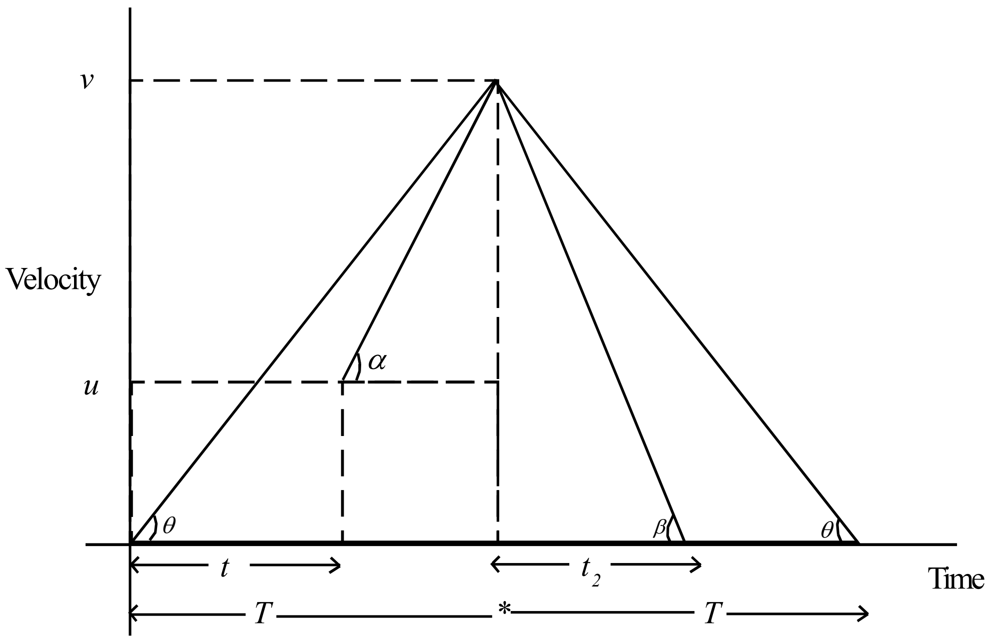
$= 23 + 25.5 = -2.5$

$Q_3 + 1.5(Q_3 - Q_1) = 40 + 1.5(40 - 23)$

$= 40 + 25.5 = 65.5$

Part B

01. (*a*)



(i) $\tan\theta = a$, , $\tan\beta = 2a$, $\tan\alpha = \dfrac{3a}{2}$

$\tan\theta = \dfrac{v}{T}$, $\quad v = aT$ ________________ (1)

$\tan\alpha = \dfrac{3a}{2} = \dfrac{v-u}{T-t}$

$2(v-u) = 3a(T-t)$ ________________ (2)

From (1) and (2)

$$2\left[aT - u\right] = 3a(T-t)$$

$$3at - 2u = aT$$

(1) $\Rightarrow$ $\quad V = 3at - 2u$

T_P = Time taken for P is $2T = 2\left(3t - \dfrac{3u}{a}\right)$

T_Q = Time taken Q is $\left(T-t\right) + t_2$

$$= 2t - \frac{2u}{a} + \frac{v}{2a}$$

$$= 2t - \frac{2u}{a} + \frac{3t}{2} - \frac{u}{a}$$

$$= \frac{7t}{2} - \frac{3u}{a}$$

(ii) Time difference $= T_P - T_Q$

$$= 2\left(3t - \frac{2u}{a}\right) - \left(\frac{7t}{2} - \frac{3u}{a}\right)$$

$$= \frac{5t}{2} - \frac{u}{a}$$

(iii) Distance travelled by P is $= \frac{1}{2}.V.2T = VT$

$$= (3at - 2u).\frac{(3at - 2u)}{a}$$

$$= \frac{(3at - 2u)^2}{a}$$

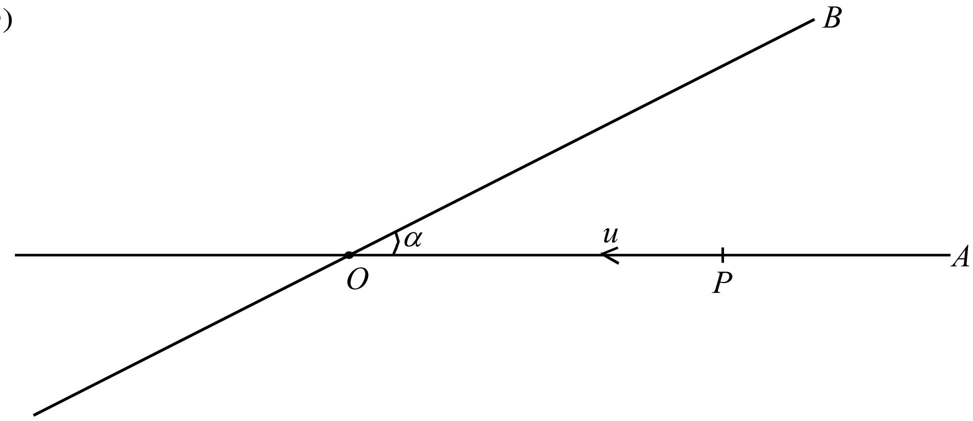
Distance travelled by Q is $= \frac{1}{2}(u + V)(T - t) + \frac{1}{2}V.t_2$

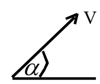
$$= \frac{1}{2}\left[u + (3at - 2u)\right]\left[2t - \frac{2u}{a}\right] + \frac{1}{2}\left[(3at - 2u)\left(\frac{3t}{2} - \frac{u}{a}\right)\right]$$

$$= \frac{1}{2}\left[(3at - u)\frac{(2at - 2u)}{a} + (3at - 2u)\frac{(3at - 2u)}{2a}\right]$$

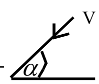
$$= \frac{1}{2a}\left[(3at - u)(2at - 2u) + \frac{(3at - 2u)^2}{2}\right]$$

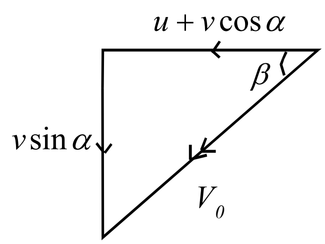
(b)



$V_{P,E} = \overleftarrow{u}$ $\leftarrow$ $V_{Q,E} =$ 

$V_{P,Q} = V_{P,E} + V_{E,Q}$

$= \underset{\leftarrow}{u} +$ 



$= \overleftarrow{u + v\cos\alpha} + \downarrow v\sin\alpha$

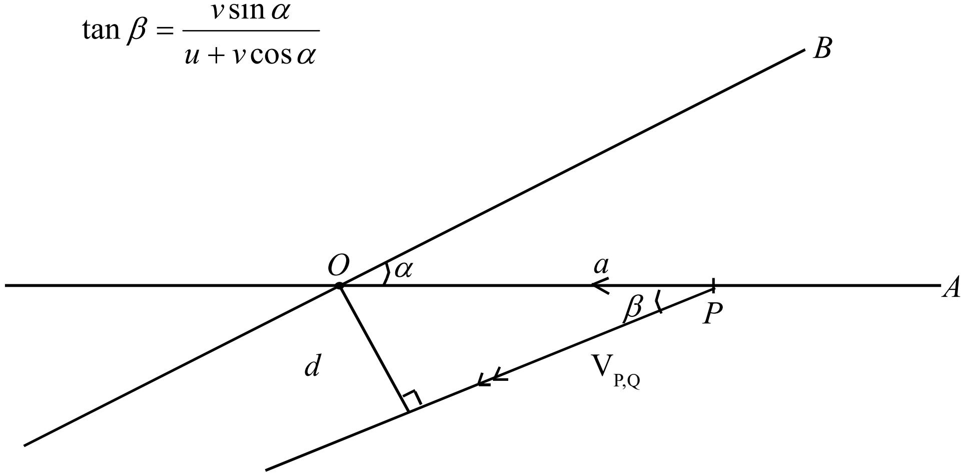
$V_0^2 = (u + v\cos\alpha)^2 + (v\sin\alpha)^2$

$V_0^2 = u^2 + v^2\cos^2\alpha + v^2\sin^2\alpha + 2uv\cos\alpha$

$V_0^2 = u^2 + v^2 + 2uv\cos\alpha$

$V_0 = \sqrt{u^2 + v^2 + 2uv\cos\alpha}$

$\tan\beta = \dfrac{v\sin\alpha}{u + v\cos\alpha}$



Shortest distance $d = a\sin\beta$

$$= \frac{av\sin\alpha}{\sqrt{u^2 + v^2 + 2uv\cos\alpha}}$$

$t =$ Time taken is $= \dfrac{PM}{V_0} = \dfrac{a\cos\beta}{V_0}$

$$t = \frac{a\left(u + v\cos\alpha\right)}{V_0^2}$$

$$t = \frac{a\left(u + v\cos\alpha\right)}{u^2 + v^2 + 2uv\cos\alpha}$$

Distance travelled by P is $= ut$

Distance travelled by Q is $= vt$

Ratio of the distances from O is $= \dfrac{a - ut}{Vt}$

$$\frac{a - \dfrac{a\left(u + v\cos\alpha\right)u}{V_0^2}}{\dfrac{va\left(u + v\cos\alpha\right)}{V_0^2}} = \frac{v + u\cos\alpha}{u + v\cos\alpha}$$

02. At the maxium speed acceleration is zero.

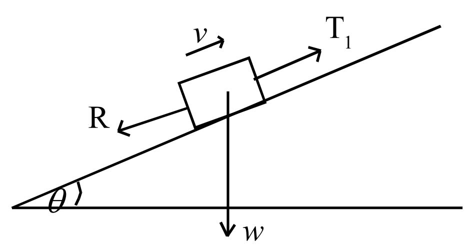
$\sin\theta = \dfrac{1}{n}$

Applying $F = ma$

$T_1 - w\sin\theta - R = \dfrac{w}{g} \times 0$

$T_1 = R + w\sin\theta$

$H = (w\sin\theta + R)v$ ———— (1)

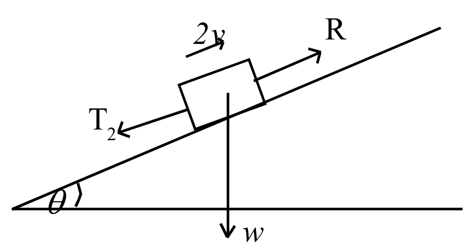


Applying $F = ma$

$T_2 + w\sin\theta - R = \dfrac{w}{g} \times 0$

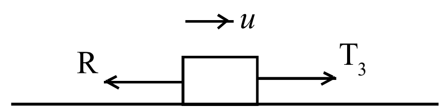
$T_2 = R - w\sin\theta$

$H = (R - w\sin\theta)2v$ ———— (2)



From (1) and (2) $R = \dfrac{3w}{n}$

Applying $F = ma$

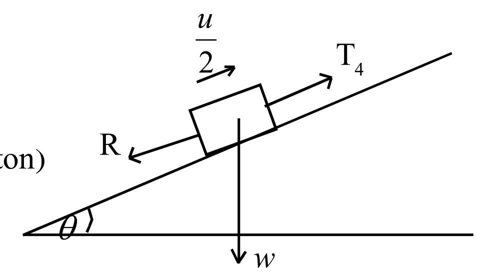


$$T_3 - R = \frac{w}{g} \times 0$$

$$T_3 = R = \frac{3w}{n}$$

$$H = T_3 . u = \frac{3uw}{n}$$

Applying $F = ma$



$$T_4 - R - w\sin\theta = \frac{w}{g} \times a \quad (a \text{ acceleraton})$$

$$T_4 = \frac{4w}{n} + \frac{wa}{g}$$

$$H = T_4 . \frac{u}{2}$$

$$\frac{u}{2}\left(\frac{4w}{n} + \frac{wa}{g}\right) = \frac{3wu}{n}$$

$$a = \frac{2g}{n}$$

(*b*) $V_{A,E} = (-3\underline{i} + 29\underline{j})$

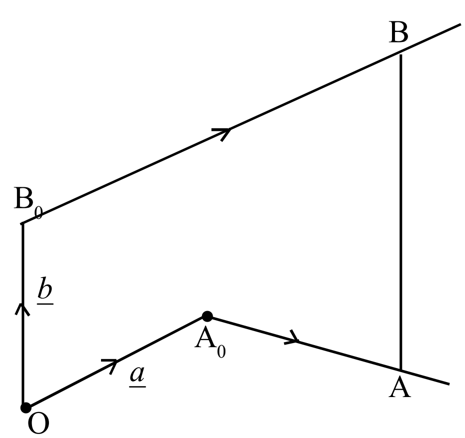
$V_{B,E} = (\underline{i} + 7\underline{j})$

$V_{B,A} = V_{B,E} + V_{E,A}$

$= V(\underline{i} + 7\underline{j}) - (-3\underline{i} + 29\underline{j})$

$V_{B,A} = (v+3)\underline{i} + (7v-29)\underline{j}$ ———————— (1)

At time t,



$\underline{r}_A = \underline{a} + \left(-3\underline{i} + 29\underline{j}\right)t$

$\underline{r}_B = \underline{b} + v\left(\underline{i} + 7\underline{j}\right)t$

$\overrightarrow{AB} = \underline{r}_B - \underline{r}_A$

$= \left[\underline{b} + v\left(\underline{i} + 7\underline{j}\right)t\right] - \left[\underline{a} + \left(-3\underline{i} + 29\underline{j}\right)t\right]$

$$\overrightarrow{AB} = (\underline{b} - \underline{a}) + (v+3)t\underline{i} + (7v-29)t\underline{j}$$

When $t = 0$, $\overrightarrow{AB} = \overrightarrow{A_0B_0} = \underline{b} - \underline{a} = [-56\underline{i} + 8\underline{j}]$

$$\overrightarrow{AB} = [-56\underline{i} + 8\underline{j}] + (v+3)t\underline{i} + (7v-29)t\underline{j}$$

$$= [(v+3)t - 56]\underline{i} + [(7v-29)t + 8]\underline{j} \quad \text{——— (2)}$$

When the particles collide $\overrightarrow{AB} = 0$

(அ-து) $(v+3)t - 56 = 0$ ——— (3)

$(7v-29)t + 8 = 0$ ——— (4)

From (3) and (4)

$v = 4$

$$\overrightarrow{AB} = [(v+3)t - 56]\underline{i} + [(7v-29)t + 8]\underline{j}$$

When $v = 3$

becomes $\overrightarrow{AB} = (6t-56)\underline{i} + (8-8t)\underline{j}$

$$\left|\overrightarrow{AB}\right| = \sqrt{(6t-56)^2 + (8-8t)^2}$$

$$= \sqrt{100(t^2 - 8t + 32)}$$

$$\left|\overrightarrow{AB}\right| = 10\sqrt{(t-4)^2 + 16}$$

AB is minimum when $t = 4$ and $\left|\overrightarrow{AB}\right|_{\min} = 40m$

When $v = 3$ and $t = 4$

$$\overrightarrow{AB} = 32\underline{i} - 24\underline{j}$$

$$\underline{V}_{A,B} = 6\underline{i} - 8\underline{j}$$

$$\underline{V}_{A,B} \cdot \overrightarrow{AB} = (6\underline{i} - 8\underline{j}) \cdot (32\underline{i} - 24\underline{j})$$

$= -192 + 192$

$= 0$

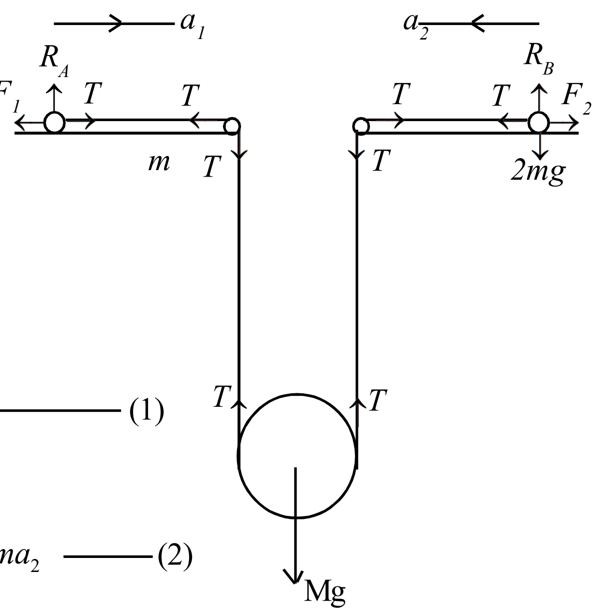
$$\underline{V}_{A,B} \cdot \overrightarrow{AB} = 0$$

Hence, $V_{A,B}$ is perpendicular to $\overrightarrow{AB}$.

03. (*a*) Let $A_{A,E} = \longrightarrow a_1$

$A_{B,E} = \longleftarrow a_2$

Then, $A_{M,E} = \downarrow \dfrac{a_1 + a_2}{2}$



When the particles are moving

$F_1 = \mu.mg,\ F_2 = \mu'(2mg)$

Applying $F = ma$

$A \rightarrow \qquad T - \mu mg = ma_1$ ——— (1)

Applying $F = ma$

$B \leftarrow \qquad T - \mu'(2mg) = 2ma_2$ ——— (2)

Applying $F = ma$

$M \downarrow \qquad Mg - 2T = M\dfrac{(a_1 + a_2)}{2}$ ——— (3)

From (1) $\quad a_1 = \dfrac{T - \mu mg}{m}$

From(2) $\quad a_2 = \dfrac{T - 2\mu' mg}{2m}$

Substituting in (3)

$$Mg - 2T = \frac{M}{2}\left[\frac{T - \mu mg}{m} - \frac{T - 2\mu' mg}{2m}\right]$$

$$Mg - 2T = \frac{MT}{2m} - \frac{\mu Mg}{2} + \frac{MT}{4m} - \frac{\mu' Mg}{2}$$

$$T\left[2 + \frac{M}{4m} + \frac{M}{2m}\right] = Mg + \frac{\mu Mg}{2} + \frac{\mu' Mg}{2}$$

$$T = \frac{2Mmg(2 + \mu + \mu')}{(3M + 8m)}$$

(ii) Given that $\mu > 2\mu'$

For the motion to take place $a_1 > 0$

$$a_1 = \frac{T}{m} - \mu g > 0$$

$$T > \mu mg$$

$$\frac{2Mmg\left(2+\mu+\mu'\right)}{\left(3M+8m\right)} > \mu mg$$

$$\frac{2+\mu+\mu'}{\mu} > \frac{3M+8m}{2M}$$

$$\frac{\mu'+2}{\mu} > \frac{3M+8m}{2m} - 1$$

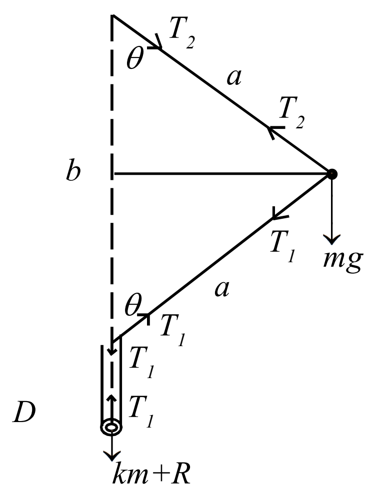
$$\frac{\mu'+2}{\mu} > \frac{M+8m}{2M}$$

$$\frac{\mu}{\mu'+2} < \frac{2M}{8m+M}$$

(*b*) Applying $F = ma$

$\uparrow B \quad T_2\cos\theta - T_1\cos\theta - mg = 0$

$\left(T_2 - T_1\right)\cos\theta = mg$ ———— (1)



Applying $F = ma$

$\overleftarrow{B} \quad \left(T_1 + T_2\right)\sin\theta = maw^2\sin\theta$

$\left(T_1 + T_2\right) = maw^2$ ———— (2)

For equilibrium of D,

$\uparrow \quad T_1 - kmg - R = 0$

$R = T_1 - kmg$ ———— (3)

$$cos\theta = \frac{b}{2a}$$

From (1) $T_2 - T_1 = \dfrac{2mga}{b}$

From (2) $T_2 + T_1 = maw^2$

$$T_1 = \frac{ma}{2}\left[w^2 - \frac{2g}{b}\right], \quad T_2 = \frac{ma}{2}\left[w^2 + \frac{2g}{b}\right]$$

From (3) $R = \frac{ma}{2}\left[w^2 - \frac{2g}{b}\right] - kmg$

$R \geq 0$

$\frac{ma}{2}\left[w^2 - \frac{2g}{b}\right] \geq kmg$

$w^2 ab \geq 2g\left(a + kb\right)$ ——————— (4)

Greatest tension in the string is λmg

$T_1, T_2 \leq \lambda mg$

$T_2 \leq \lambda mg$

$\frac{ma}{2}\left[w^2 + \frac{2g}{b}\right] \leq \lambda mg$

$w^2 \leq \frac{2\lambda g}{a} - \frac{2g}{b}$

From (4) $w^2 \geq \frac{2g}{b} + \frac{2kg}{a}$

$\frac{2g}{b} + \frac{2kg}{a} \leq w^2 \leq \frac{2\lambda g}{a} - \frac{2g}{b}$

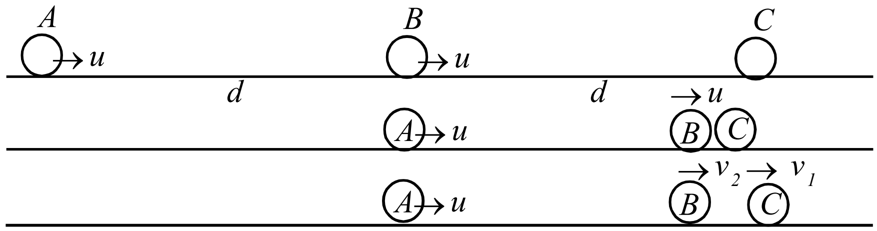
$\frac{2g}{b} + \frac{2kg}{a} \leq \frac{2\lambda g}{a} - \frac{2g}{b}$

$\frac{1}{b} + \frac{k}{a} \leq \frac{\lambda}{a} - \frac{1}{b}$

$\frac{2}{b} \leq \frac{\lambda - k}{a}$

$\left(\lambda - k\right)b \geq 2a$

04. (a)



First collission (B and C)

Using $I = \Delta mv$ for the system

$\rightarrow m(v_2 - u) + m(v_1 - 0) = 0$

$\therefore mv_1 + mv_2 = mu$

$v_1 + v_2 = u$ ——————— (1)

Newton's experimental law,

$v_1 - v_2 = eu$ ——————— (2)

From (1) and (2) $v_1 = \frac{u}{2}(1+e), \quad v_2 = \frac{u}{2}(1-e)$

Time taken for A to collide with B (say t_0)

$$t_0 = \frac{d}{u} + \frac{d}{u - v_2}$$

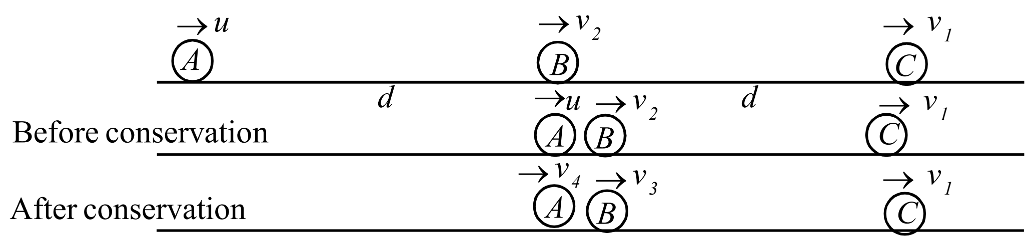
$$= \frac{d}{u} + \frac{d}{u - \frac{u}{2}(1-e)}$$

$$= \frac{d}{u} + \frac{2d}{u(1+e)}$$

$$\frac{d(3+e)}{u(1+e)}$$

Distance travelled by A is $= ut_0$

$$= \frac{d(3+e)}{(1+e)}$$



Second collission (A and B)

Using $I = \Delta mv$ for the system

$\rightarrow \quad m(v_4 - u) + m(v_3 - v_2) = 0$

$$v_3 + v_4 = u + v_2 \qquad \text{———} \quad (3)$$

Newton's experimental law,

$$v_3 - v_4 = e(u - v_2) \qquad \text{———} \quad (4)$$

From (3) and (4) $v_3 = \frac{u}{2}(1+e) + \frac{v_2}{2}(1-e)$

$$= \frac{u}{2}(1+e) + \frac{u}{4}(1-e)^2$$

$$= \frac{u}{4}\left[2 + 2e + 1 - 2e + e^2\right]$$

$$v_3 = \frac{u}{4}\left[3 + e^2\right]$$

Now $\qquad v_3 - v_1 = \frac{u}{4}\left[3 + e^2\right] - \frac{u}{2}[1+e]$

$$= \frac{u}{4}\left[1 - 2e + e^2\right]$$

$$= \frac{u}{4}[1-e]^2$$

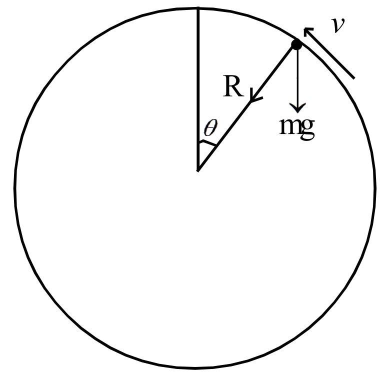
$$v_3 > v_1$$

Hence there will be another collision between A and B.

(b) Law of conservation of energy

$$\frac{1}{2}mu^2 + 0 = \frac{1}{2}mv^2 + mga(1 + cos\theta)$$

$$v^2 = u^2 - 2ag(1 + cos\theta) \qquad \text{———} \quad (1)$$



$\swarrow$ $F = ma$

$$R + mgcos\theta = \frac{mv^2}{a} \quad \text{——— (2)}$$

$$R = \frac{m}{a}\left[u^2 - 2ag - 3agcos\theta\right]$$

When $R = 0$, $\theta = \alpha$ (say)

$$O = u^2 - 2ag - 3agcos\alpha$$

$$cos\alpha = \frac{u^2 - 2ag}{3ag}$$

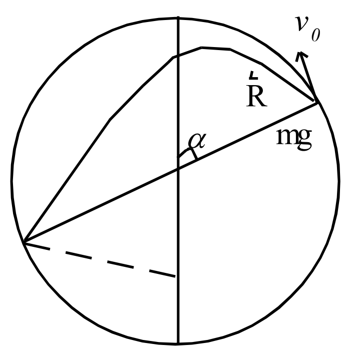
$2ag < u^2 < 5ag$ (Given)

$O < cos\alpha < 1$

Hence α is an acute angle.

hence the particle leaves the sphere before it reaches the highest point

when $cos\alpha = \dfrac{u^2 - 2ag}{3ag}$



When $\theta = \alpha$, $v = v_0$ (say)

From (1) $v_0{}^2 = u^2 - 2ag\left(1 + cos\alpha\right)$

$$= u^2 - 2ag - 2agcos\alpha$$

$$= 3agcos\alpha - 2agcos\alpha$$

$$v_0{}^2 = agcos\alpha \quad \text{——— (3)}$$

$$S = ut + \frac{1}{2}at^2$$

$\longleftarrow$ $2a\sin\alpha = v_0 c\,\mathrm{os}\,\alpha . t_0$ ——— (4)

$\uparrow$ $-2a\cos\alpha = v_0 c\,\mathrm{os}\,\alpha . t_0 - \dfrac{1}{2}gt_0{}^2$ ——— (5)

From (4) and (5)

$$-2a\cos\alpha = \frac{2a\sin^2\alpha}{\cos\alpha} - \frac{2a^2 g}{v_0{}^2} . \frac{\sin^2\alpha}{\cos^2\alpha}$$

$$\frac{a^2 g}{v_0{}^2} . \frac{\sin^2\alpha}{\cos^2\alpha} = \frac{a}{\cos\alpha}$$

$$v_0^{\ 2} = \frac{ag\sin^2\alpha}{\cos\alpha} \quad \text{———————} \quad (6)$$

From (3) and (6)

$$\tan^2\alpha = 1$$

$$\alpha = 45^0$$

$$u^2 - 2ag = 3ag\cos 45^0$$

$$u^2 = \left(\frac{3}{\sqrt{2}} + 2\right)ag$$

05. (a) Let the time of light be t எனக.

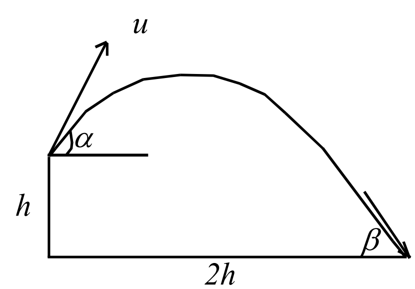
$$S = ut + \frac{1}{2}at^2 \quad \text{———————} \quad (1)$$

$$\longrightarrow 2h = u\cos\alpha.t \quad \text{———————} \quad (2)$$

$$\uparrow \; -h = u\sin\alpha.t - \frac{1}{2}gt^2$$

From (1)
Substituting in (2)

$$-h = u\sin\alpha.\frac{2h}{u\cos\alpha} - \frac{1}{2}.g\frac{4h^2}{u^2\cos^2\alpha}$$



$$-1 = 2\tan\alpha - \frac{2gh}{u^2\cos^2\alpha}$$

$$1 + 2\tan\alpha = \frac{2gh}{u^2\cos^2\alpha}$$

$$u^2\cos^2\alpha = \frac{2gh}{1 + 2\tan\alpha}$$

$$u^2 = \frac{2gh\left(1 + \tan^2\alpha\right)}{\left(1 + 2\tan\alpha\right)}$$

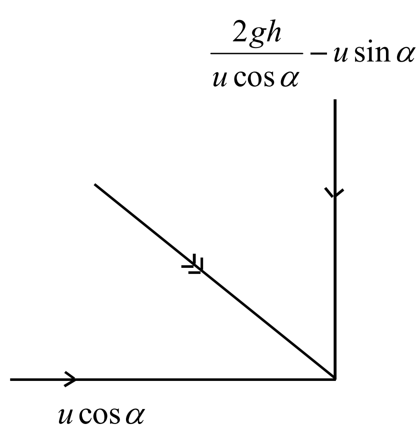
$$v = u + at$$

$$\uparrow \quad v_1 = u\sin\alpha - gt$$

$$= u\sin\alpha - g \times \frac{2h}{u\cos\alpha}$$

$$v_1 = u\sin\alpha - \frac{2gh}{u\cos\alpha}$$

$\downarrow$ Velocity $= \dfrac{2gh}{u\cos\alpha} - u\sin\alpha$

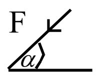


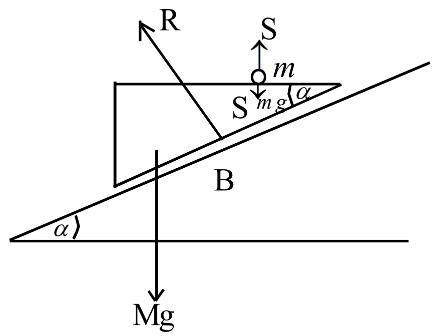
$$\tan\beta = \frac{\dfrac{2gh}{u\cos\alpha} - u\sin\alpha}{u\cos\alpha}$$

$$\tan\beta = \frac{2gh}{u^2\cos^2\alpha} - \tan\alpha$$

$$= 1 + 2\tan\alpha - \tan\alpha$$

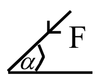
$$\tan\beta = 1 + \tan\alpha$$

(b) $\quad A_{m,E} =$ 



$A_{m,M} = \longrightarrow f$

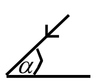
$A_{m,E} = A_{m,M} + A_{M,E}$

$\longrightarrow f +$ 

Applying $F = ma$

(M, m) System

$\leftarrow,\ R\sin\alpha = MF\cos\alpha + m(F\cos\alpha - f)$ ——— (1)



$(M + m)g\sin\alpha = MF + m(F - f\cos\alpha)$ ——— (2)

m for $\leftarrow F = ma$

$0 = m(F\cos\alpha - f)$ ——— (3)

From (3) $f = F\cos\alpha$

Substituting in (2),

$$\left(M + m\right)g\sin\alpha = MF + m(F - F\cos^2\alpha)$$

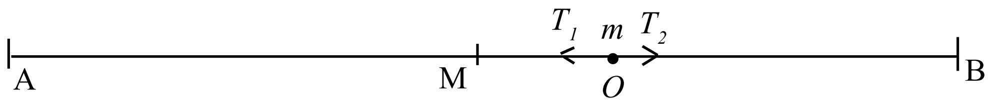
$$\left[M + m\sin^2\alpha\right]F = \left(M + m\right)g\sin\alpha$$

$$F = \frac{\left(M + m\right)g\sin\alpha\cos\alpha}{\left(M + m\sin^2\alpha\right)}$$

$$f = \frac{M(M+m)g\cos\alpha}{(M+m\sin^2\alpha)}$$

From (1), $R = \dfrac{M(M+m)g\cos\alpha}{(M+m\sin^2\alpha)}$

06.



$AM = MB = 4l$, Let $MO = d$.

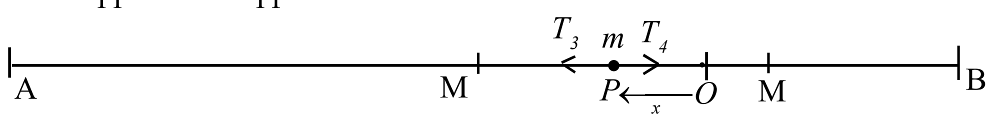
$O \rightarrow \quad T_1 = T_2$

$$\frac{\lambda(d+2l)}{2l} = \frac{4\lambda(l-d)}{3l}$$

$$3(d+2l) = 8(l-d)$$

$$11d = 2l$$

$$d = \frac{2l}{11}, \quad OM = \frac{2l}{11}$$



Let $OP = x$

$\underline{F} = ma$

$\leftarrow T_3 - T_4 = m\ddot{x}$

$$\frac{\lambda}{2l}\left[\left(4l + \frac{2l}{11} - x\right) - 2l\right] - \frac{4\lambda}{3l}\left[\left(4l - \frac{2l}{11} + x\right) - 3l\right] = m\ddot{x}$$

$$\frac{\lambda}{2l}\left[\frac{24l}{11} - x\right] - \frac{4\lambda}{3l}\left[\frac{9l}{11} + x\right] = m\ddot{x}$$

$$\ddot{x} = \frac{11\lambda}{6ml}x$$

Hence, motion is S.H.M.

(i) Centre of oscillation is $x = O$, (i.e) O

$$V^2 = \frac{11\lambda}{6ml}\left[A^2 - x^2\right] \quad (A = \text{amplitude})$$

When $x = \dfrac{2l}{11}$, $v = 0$

Therefore $A = \dfrac{2l}{11}$

Hence, when the particle comes to M', where

$OM' = \frac{2l}{11}$, it instantance onsly comes to rest and

$$BM' = 4l - \frac{4l}{11} = \frac{40l}{11} > 3l$$

The string is always tant.

The period of oscillation is $\frac{2\pi}{\omega}\left(\omega^2 = \frac{11\lambda}{6ml}\right)$

$$= 2\pi\sqrt{\frac{6ml}{11\lambda}}$$

$$V^2 = \frac{11\lambda}{6ml}\left[\left(\frac{2l}{11}\right)^2 - x^2\right]$$

$$MC = \frac{3l}{11}, \quad OC = \frac{3l}{11} - \frac{2l}{11} = \frac{l}{11}$$

When $x = \frac{l}{11}$, Let $v = v_0$

$$V_0^2 = \frac{11\lambda}{6ml}\left[\left(\frac{2l}{11}\right)^2 - \left(-\frac{l}{11}\right)^2\right]$$

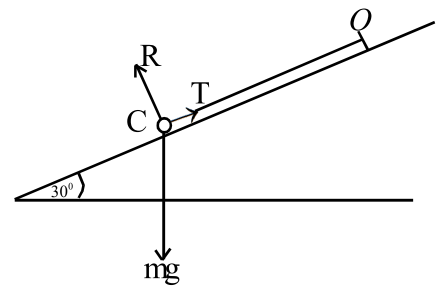
$$V_0^2 = \frac{11\lambda}{6ml} \times \frac{3l^2}{11 \times 11}$$

$$V_0^2 = \frac{\lambda l}{22m}$$

$$V_0^2 = \sqrt{\frac{\lambda l}{22m}}$$

07. Let $OC = d$

For equilibrium of m



$T - mg\sin 30^0 = 0$

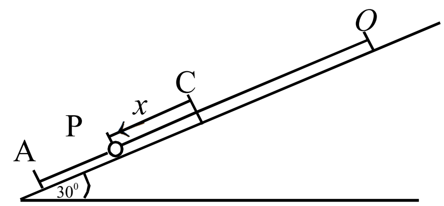
$2T = mg$

$2 \times \dfrac{3mg(d-6a)}{6a} = mg$

$d = 7a$

Let $CA = 2a$ and $CP = x$

Energy at A



$= 0 - mg.2a.\sin 30^0 + \dfrac{1}{2}.mg \times \dfrac{(3a)^2}{6a}$

Energy at P

$-\dfrac{1}{2}m\dot{x}^2 - mgx\sin 30^0 + \dfrac{1}{2} \times 3mg \times \dfrac{(a+x)^2}{6a}$

Principle of conservation of energy

$= 2mga.\sin 30^0 + \dfrac{mg}{4a} \times 9a^2 = \dfrac{1}{2}m\dot{x}^2 - \dfrac{mgx}{2} + \dfrac{mg}{4a}(a+x)^2$

Differentiating *w.r.t.* time t

$O = \dfrac{1}{2}m2\dot{x}\ddot{x} - \dfrac{mg\dot{x}}{2} + \dfrac{mg}{4}.2(a+x)\dot{x}$

$O = \ddot{x} - \dfrac{g}{2} + \dfrac{g}{2a}(a+x)$

$\ddot{x} + \dfrac{g}{2a}x = 0$

$x = A\cos\omega t + B\cos\omega t$ $\qquad \left(\omega^2 = \dfrac{g}{2a}\right)$

$v = \dfrac{dx}{dt} = \dot{x} = -A\omega\sin\omega t + B\omega\sin\omega t$

$t = 0$, $x = 2a$ and $\dot{x} = 0$

$2a = A$ ——— (1)

$0 = 0 + B\omega$ ——— (2)

$B = 0$

$x = 2a\cos\omega t$

The string becomes slack when $x = -a$

Let $x = -a$ when $t = t_1$

$$-a = 2a\cos\omega t_1$$

$$\cos\omega t_1 = -\frac{1}{2}$$

$$\omega t_1 = \frac{2\pi}{3}$$

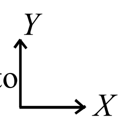
$$t_1 = \frac{1}{\omega}.\frac{2\pi}{3}$$

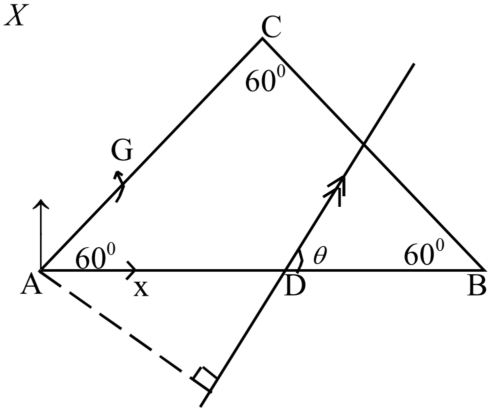
$$t_1 = \frac{2\pi}{3}\sqrt{\frac{2a}{g}}$$

when $\omega t_1 = \dfrac{2\pi}{3}$, $\dot{x} = 2a\omega\sin\omega t$

$$\dot{x} = 2a\sqrt{\frac{2a}{g}}\sin\frac{2\pi}{3}$$

$$\dot{x} = -\sqrt{\frac{3ag}{2}}, \qquad \text{Speed is } \sqrt{\frac{3ag}{2}}$$

08. (a) Suppose that the system reduces to  and a couple $G\curvearrowright$ at A



$A\circlearrowleft$, $\quad M = G$

$B\circlearrowleft$, $\quad \dfrac{M}{2} = -Y.2a + G$

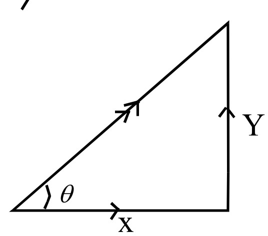
$C\circlearrowleft$, $\quad 2M = X.\sqrt{3}a - Y.a + G$

$$G = M,\ Y = \frac{M}{4a},\ X = \frac{5M}{4\sqrt{3}a}$$

$$R = \sqrt{X^2 + Y^2} = \frac{M}{a}\sqrt{\frac{1}{16} + \frac{25}{48}}$$

$$R = \frac{M}{a}\sqrt{\frac{7}{12}}$$

$$\tan\theta = \frac{Y}{X} = \frac{5}{\sqrt{3}}$$



Taking moment about A,

$R.AD\sin\theta = M$

$\left(R\sin\theta\right)AD = M$

$Y.AD = M$

$AD = \dfrac{M}{Y} = 4a$

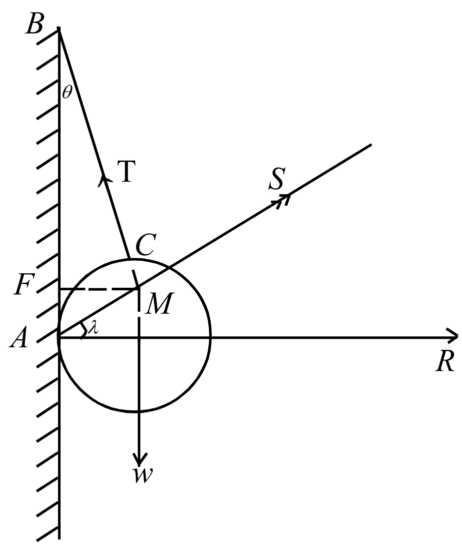
Forces acting on the sphere

O at W

C at T

Resultant of F and R is S

Now three forces T, W, S meet at a point M.



$AB = h$, $OA = a$, $O\hat{A}M = \lambda$

Where $\mu = \tan\lambda$

$OM = a\tan\lambda = a\mu$

$\tan\theta = \dfrac{a}{h - OM} = \dfrac{a}{h - a\mu}$

$\theta = \tan^{-1}\left(\dfrac{a}{h - a\mu}\right)$

When $\mu = \dfrac{h}{2a}$, $\theta = \tan^{-1}\left(\dfrac{a}{2a\mu - a\mu}\right)$

$\theta = \tan^{-1}\left(\dfrac{1}{\mu}\right)$

$\tan\theta = \dfrac{1}{\mu}$

Consider the tringle AMB

$\left.\begin{array}{l} T \to MB \\ W \to BA \\ S \to AM \end{array}\right\}$ AMB is the triangle of forces.

$$\frac{T}{\sin(90-\lambda)} = \frac{W}{\sin\left[90-(\theta-\lambda)\right]} = \frac{S}{\sin\theta}$$

$$\frac{T}{\cos\lambda} = \frac{W}{\cos(\theta-\lambda)}$$

$$T = \frac{W\cos\lambda}{\cos(\theta-\lambda)}$$

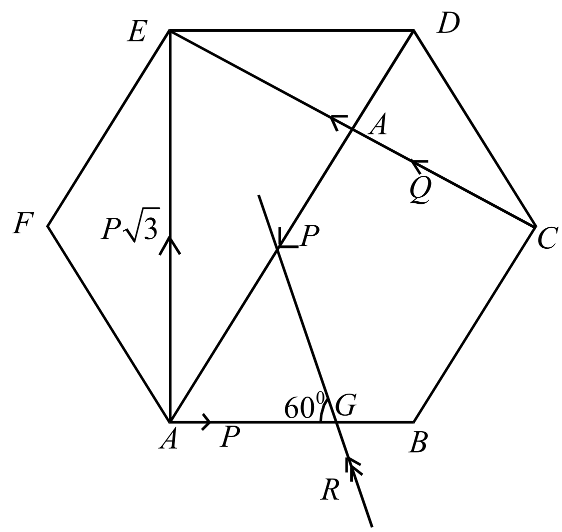
$$T = \frac{W\cos\lambda}{\cos\theta\cos\lambda + \sin\theta\sin\lambda}$$

$$T = \frac{W}{\cos\theta + \sin\theta\tan\lambda}$$

$$T = \frac{W}{\dfrac{\mu}{\sqrt{1+\mu^2}} + \dfrac{\mu}{\sqrt{1+\mu^2}}}$$

$$T = \frac{W\sqrt{1+\mu^2}}{2\mu}$$

09. (a) $\longleftarrow$



$$X = P - P\cos 60^0 - Q\cos 30^0$$

$$X = \frac{P - Q\sqrt{3}}{2}$$

$$\uparrow\ Y = P\sqrt{3} - P\sin 60^0 + Q\sin 30^0$$

$$Y = \frac{Q + P\sqrt{3}}{2}$$

(i) If the system reduces to couple

$X = 0$, and $Y = 0$

Then $P = Q\sqrt{3}$ and $Q = 0$

But $Q \neq 0$

The system cannot reduce to a couple.

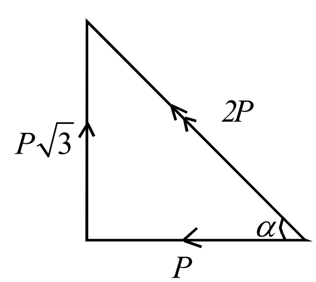
(ii) If $Q = P\sqrt{3}$

$X = -P$, $Y = P\sqrt{3}$

$\therefore\ R^2 = P^2 + \left(\sqrt{3}P\right)^2$

$R = 2P$

$\tan\alpha = \sqrt{3}$, $\alpha = 60^0$



(iii) Moments about A

Moment of the resultant about A = Moment of the forces about A

$$R.AG\sin 60^0 = Q.\frac{3a}{2}$$

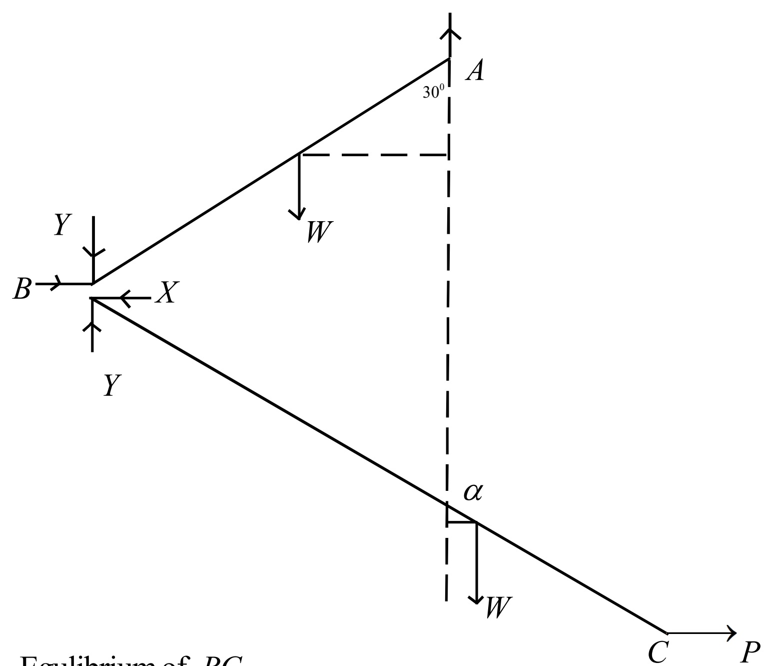
$$\left(R\sin 60^0\right)AG = P\sqrt{3}.\frac{3a}{2}$$

$$Y.AG = \frac{3\sqrt{3}Pa}{2}$$

$$AG = \frac{3\sqrt{3}Pa}{2}\times\frac{1}{\sqrt{3}P}$$

$$AG = \frac{3a}{2}$$

(b)



Egulibrium of BC

B↻ $-Wa\sin\alpha + P2a\cos\alpha = 0$

$$P = \frac{W}{2}\tan\alpha$$

$\longrightarrow$ $P - X = 0 \quad X = P$

$\uparrow$ $Y - W = 0 \quad Y = W$

For equilibrium of AB,

$A\circlearrowleft \quad Wa\sin 30^0 + Y.2a\sin 30^0 - X.2a\cos 30^0 = 0$

$$\frac{W}{2} + W - P\sqrt{3} = 0$$

$$P = \frac{\sqrt{3}W}{2}$$

Reaction at B is

$$R = \sqrt{X^2 + Y^2}$$

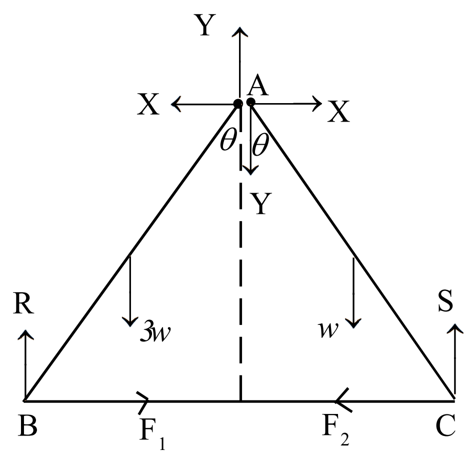
$$R = \sqrt{\frac{3W^2}{4} + W^2} \qquad \tan\alpha = \frac{2P}{W} = \sqrt{3}$$

$$R = \sqrt{\frac{7W}{2}} \qquad \alpha = 60^0$$

$$\tan\theta = \frac{Y}{X} = \frac{2}{\sqrt{3}}$$

$$\theta = \tan^{-1}\left(\frac{2}{\sqrt{3}}\right)$$

10. (a) For equilibrium of AB and AC



$\uparrow \quad R + S - 4w = 0$

$$R + S = 4w$$

$B\circlearrowleft = 0$

$$S.4a\sin\theta - w.3a\sin\theta - 3w.a\sin\theta = 0$$

$$S = \frac{3w}{2}, \quad R = \frac{5w}{2}$$

$\longrightarrow \quad F_1 - F_2 = 0; \quad F_2 = F_2 \quad (=F, \text{ say})$

For equilibrium of AB, $A = 0$

$$F.2a\cos\theta - R.2a\sin\theta + 3w.a\sin\theta = 0$$

$$F = w\tan\theta$$

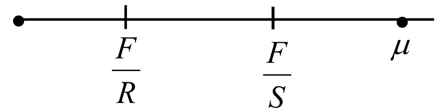
$$\frac{5w}{2} > \frac{3w}{2}$$

$$R > S$$

$$\frac{1}{R} < \frac{1}{S}$$

$$\frac{F}{R} < \frac{F}{S}$$

For equilibrium , $\frac{F}{R} \leq \mu$, $\frac{F}{S} \leq \mu$



i.e $\frac{F}{R} < \frac{F}{S} \leq \mu$

When θ increases $\frac{F}{S}$ reaches μ first and

Limiting occurs at C first.

Now $\frac{F}{R} = \frac{w \tan\theta \times 2}{5w} = \frac{2\tan\theta}{5}$

$\frac{F}{S} = \frac{w \tan\theta \times 2}{3w} = \frac{2\tan\theta}{3}$

Hence $\frac{F}{S} \leq \mu$

$2\frac{\tan\theta}{3} \leq \mu$

$\tan\theta \leq \frac{3\mu}{2}$

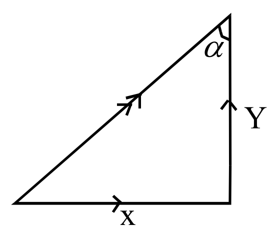
For AB

$\longrightarrow$ $F - X = 0$

$X = F = w\tan\theta$

$\uparrow$ $Y + R - 3w = 0$

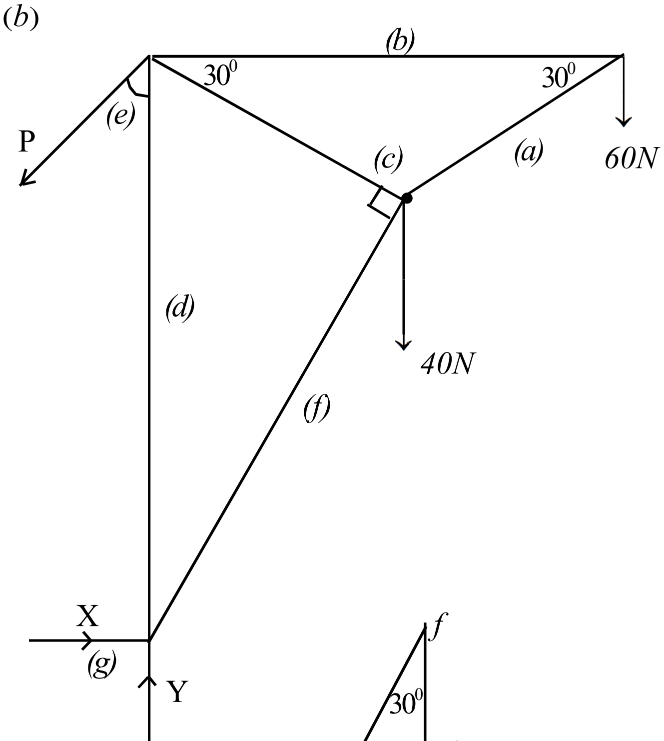
$Y = \frac{w}{2}$



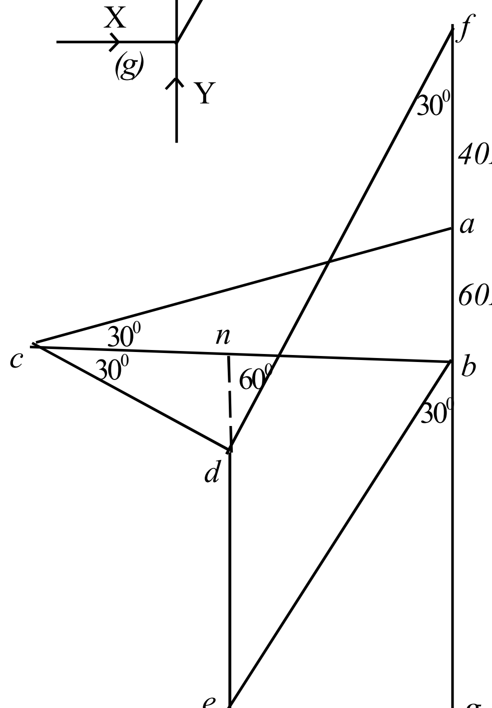
$\tan\alpha = \frac{X}{Y} = 3\mu$

$\alpha = \tan^{-1}(3\mu)$

(b)



Rod	Thrust	Tension
AB	100	-
BC	-	$60\sqrt{3}$
CD	120	-
DB	40	-
AD	$80\sqrt{3}$	-

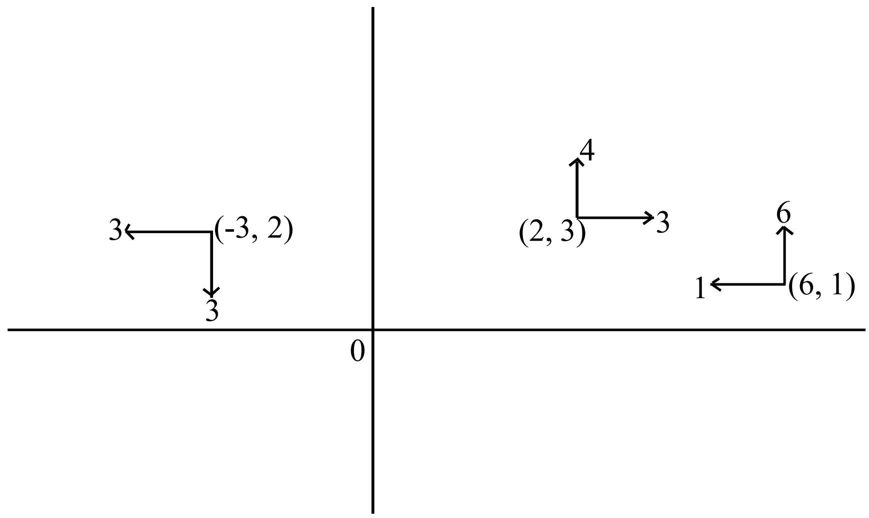


$$X = 40\sqrt{3}$$
$$Y = 220$$

Resultant at A

$$R = \sqrt{X^2 + Y^2}$$
$$= \sqrt{\left(40\sqrt{3}\right)^2 + 220^2}$$
$$= 20\sqrt{133}N$$
$$\tan\alpha = \frac{Y}{X}$$
$$\tan\alpha = \frac{220}{40\sqrt{3}}$$
$$\tan\alpha = \frac{11}{2\sqrt{3}}$$

11. (*a*)



Resultant $\underline{R} = \underline{F_1} + \underline{F_2} + \underline{F_3}$

$$= (3i + 4j) + (i + 6j) + (3i - 3j)$$

$\underline{R} = -\underline{i} + 7\underline{j}$

$X = -1,\ Y = 7$ $\qquad |\underline{R}| = \sqrt{1^2 + 7^2} = 5\sqrt{2}N$

$O\circlearrowleft = M = (9 + 6) + (36 + 1) + (8 - 9) = 15 + 37 - 1 = 51$

O Taking moment about $=$ Algebraic sum of the moments of the forces about O

$Y.x - X.y = M$

$7x + y = 51$

Equation of line of action is $7x + y - 51 = 0$

For equilibrium O $\underline{F_4} = \underline{i} - 7\underline{j}$ and $G = -51$

(*b*) $A\circlearrowleft = \lambda BC.h_1 = \lambda \times \frac{1}{2} \times 2BC \times h_1$

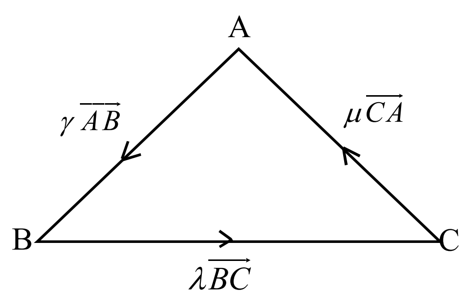
$$= 2\lambda.\Delta ABC$$

$B\circlearrowleft = \mu CA..h_2 = \mu \times 2 \times \frac{1}{2} \times CA \times h_2$

$$= 2\mu.\Delta ABC$$

$C\circlearrowleft = \gamma AB..h_3 = \gamma \times 2 \times \frac{1}{2} \times AB \times h_3$

$$= 2\gamma.\Delta ABC$$



[h_1, h_2 and h_3 are the perpendicular distances from *A, B* and C to *BC, CA, AB* respectively. $\Delta ABC =$ Area of the triangle *ABC*]

(i) Suppose that $\lambda = \mu = \gamma$

$\overset{\curvearrowright}{A} = \overset{\curvearrowright}{B} = \overset{\curvearrowright}{C} \neq 0$

Since the moments about three points not in a straight line are constant and not equal to zero, the system reduces to a couple.

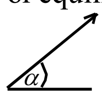
(ii) Conversely assume that the system redues to a couple.

$\therefore \quad \overset{\curvearrowright}{A} = \overset{\curvearrowright}{B} = \overset{\curvearrowright}{C}$

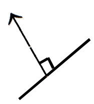
$2\lambda.\Delta ABC = 2\mu\Delta ABC = 2\gamma.\Delta ABC$

$\therefore \quad \lambda = \mu = \gamma$

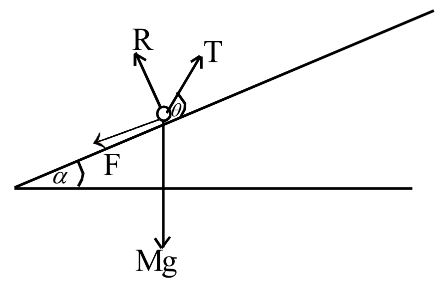
(c) For equilibrium of M,



$$T\cos\theta - F - Mg\sin\alpha = 0$$



$$R + T\sin\theta - Mg\cos\alpha = 0$$



At limiting, $\dfrac{F}{R} = \mu$

$$\frac{T\cos\theta - Mg\sin\alpha}{Mg\cos\alpha - T\sin\theta} = \frac{\sin\lambda}{\cos\lambda}$$

$$T\cos(\theta - \lambda) = Mg\sin(\alpha + \lambda)$$

$$T = \frac{Mg\sin(\alpha + \lambda)}{\cos(\theta - \lambda)} \qquad \text{——————} \quad (1)$$

For T to be minimum $\cos(\theta - \lambda)$ should be maximum

$$\cos(\theta - \lambda) = 1$$

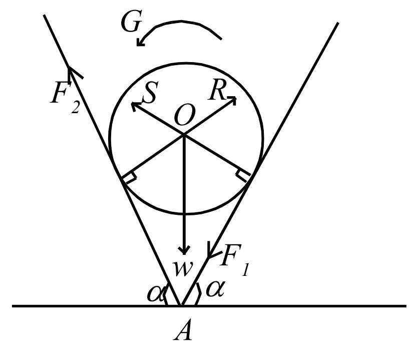
$$\theta = \lambda$$

$$T_{\text{mini}} = Mg\sin(\alpha + \lambda)$$

Hence to find the least force acting parallel to the place, put $\theta = 0$ in (1)

Hence the require force is $= \dfrac{Mg\sin(\alpha + \lambda)}{\cos(-\lambda)}$

$$= \frac{P}{\cos\lambda} = P\sec\lambda$$



12 (*a*) For equilibrium mements about O

$$O\circlearrowleft \quad G - F_1.a - F_2.a = 0$$

$$G = \left(F_1 + F_2\right)a$$

At limiting equilibrium

$$F_1 = \mu S\,,\quad F_2 = \mu R$$

$$G = \mu a\left(R + S\right) \text{ ——— } (1)$$

$$A\circlearrowleft \quad S.a\tan\alpha - R.a\tan\alpha + G = 0$$

$$G = a\tan\alpha(R - S) \text{ ——— } (2)$$

$$\uparrow \quad (R + S)\cos\alpha + F_2\sin\alpha - F_1\sin\alpha - w = 0$$

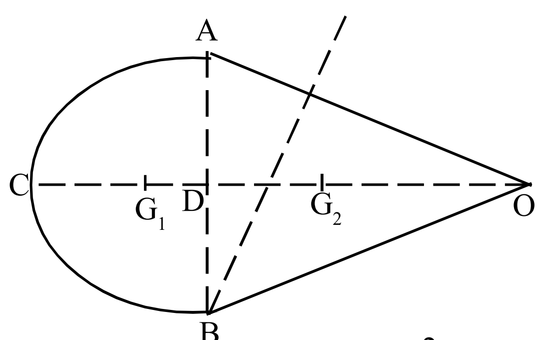
$$(R + S)\cos\alpha + \mu\sin\alpha(R - S) - w = 0 \text{ ——— } (3)$$

From (1) and (2) $\dfrac{G\cos\alpha}{\mu a} + \mu\sin\alpha\dfrac{G}{a\tan\alpha} - w = 0$

$$\frac{G\cos\alpha}{a}\left(\frac{1}{\mu} + \mu\right) = w$$

$$G = \frac{\mu a w}{\left(1 + \mu^2\right)\cos\alpha}$$

(*b*) By symmetry centre of granity lies on OC.



Mass of the hemi sphere $M_1 = \dfrac{2}{3}\pi r^3\sigma\,,\quad DG_1 = \dfrac{3r}{8}$

Mass of the cone $M_2 = \dfrac{1}{3}\pi r^2 \times 4r \times \rho = \dfrac{4}{3}\pi r^3\rho$

$$DG_2 = \frac{1}{4}\times 4r = r$$

Total mass of the composite body is $\left(M_1 + M_2\right)$

Let $DG = \overline{x}$

$D\circlearrowleft$ $\left(M_1 + M_2\right)\overline{x} = M_2.DG_2 - M_1.DG_1$

$$\left(\frac{2}{3}\pi r^3\sigma + \frac{4}{3}\pi r^3\rho\right)\overline{x} = \frac{4}{3}\pi r^3\rho r - \frac{2}{3}\pi r^3\sigma \times \frac{3r}{8}$$

$$\frac{2}{3}\pi r^3\left(\sigma + 2\rho\right)\overline{x} = \frac{4}{3}\pi r^4\left(\rho - \frac{3\sigma}{8}\right)$$

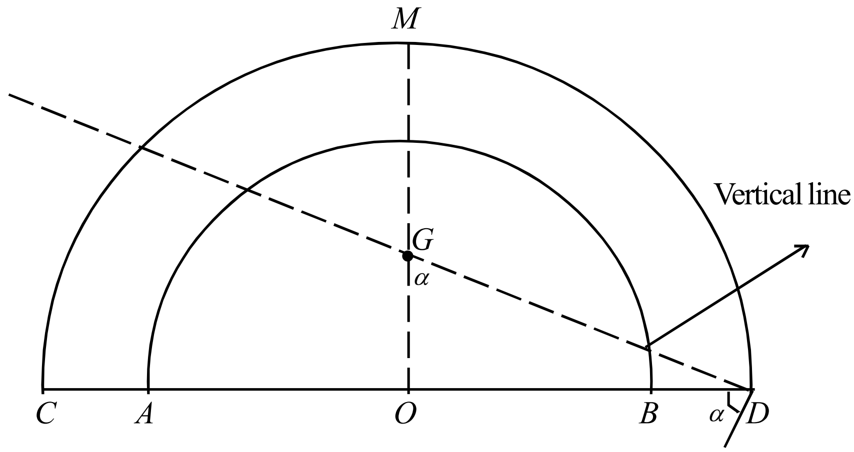
$$\overline{x} = \frac{r}{8}\frac{\left(16\rho - 3\sigma\right)}{\left(\sigma + 2\rho\right)}$$

If $\rho = \sigma$, $\quad \overline{x} = \dfrac{13r}{24}$

$$\tan\theta = \frac{r}{\overline{x}} = \frac{24}{13}$$

$$\theta = \tan^{-1}\left(\frac{24}{13}\right)$$

13.



By symmetry centre of gravity lies on OM

Solid	Mass	Centre of gravity
Hemisphere CMD	$M_1 = \frac{2}{3}\pi(2a)^3\rho$	$OG_1 = \frac{3}{8}\times 2a = \frac{3a}{4}$
Hemisphere ALB	$M_2 = \frac{2}{3}\pi a^3\rho$	$OG_2 = \frac{3a}{8}$
Bowl CD	$M_1 - M_2 = \frac{14}{3}\pi a^3\rho$	OG

$O\circlearrowright$ $\quad (M_1 - M_2)OG = M_1.OG_1 - M_2.OG_2$

$$\frac{14}{3}\pi a^3\rho OG = \frac{16}{3}\pi a^3 \times \frac{3a}{4} - \frac{2}{3}\pi a^3 \times \frac{3a}{8}$$

$$OG = \frac{45a}{56}$$

$$\tan\alpha = \frac{2a}{OG} = \frac{112}{45}$$

$$\alpha = \tan^{-1}\left(\frac{112}{45}\right)$$

At the point of toppling

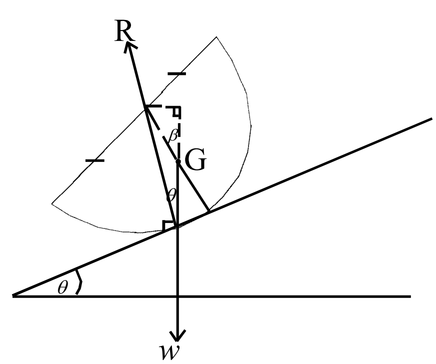
$a\sin\theta = OGSn\beta \le OG$

$a\sin\theta \le OG$

$$\sin\theta \le \frac{45a}{56a}$$

$$\sin\theta \le \frac{45}{56}$$

$$\theta \le \sin^{-1}\left(\frac{45}{56}\right)$$

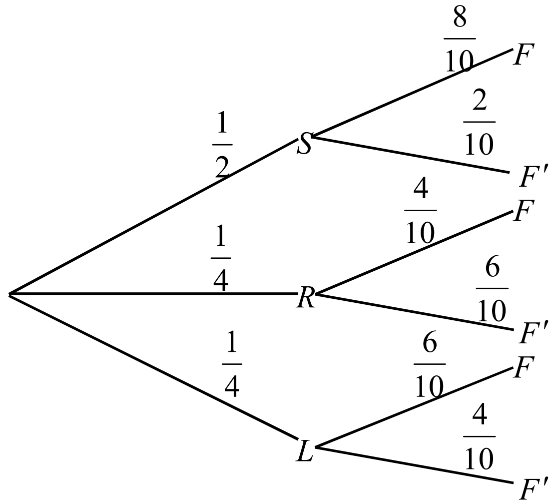


14. (*a*) Use the following notation.

S: he goes to sea.
R: he goes to the river.
L: he goes to the lake.
F: he catches fish.

$P(S)=\frac{1}{2},\quad P(R)=\frac{1}{4},\quad P(L)=\frac{1}{4}$

$P(\mathrm{F}\mid\mathrm{S})=\frac{8}{10},\quad P(\mathrm{F}\mid\mathrm{R}))=\frac{4}{10},\quad P(F\mid L)=\frac{6}{10}$



$P(S\cap F)=\frac{1}{2}\times\frac{8}{10}$

$P(S\cap F')=\frac{1}{2}\times\frac{2}{10}$

$P(R\cap F)=\frac{1}{4}\times\frac{4}{10}$

$P(R\cap F')=\frac{1}{4}\times\frac{6}{10}$

$P(L\cap F)=\frac{1}{4}\times\frac{6}{10}$

$P(L\cap F')=\frac{1}{4}\times\frac{4}{10}$

(i) Using the law of total probability,

$$P(F)=\frac{1}{2}\times\frac{8}{10}+\frac{1}{4}\times\frac{4}{10}+\frac{1}{4}\times\frac{6}{10}$$

$$P(F)=\frac{13}{20}$$

(ii) $P(\text{Catches fish on 2 sundays}) \quad = {}^3C_2\times\left(\frac{13}{20}\right)^2\times\frac{7}{20}$

$P(\text{Catches fish on all 3 sundays}) \quad = {}^3C_3\times\left(\frac{13}{20}\right)^3$

The required probability is $\quad = {}^3C_2\times\left(\frac{13}{20}\right)^2\times\frac{7}{20}+{}^3C_3\times\left(\frac{13}{20}\right)^3$

$$=\frac{2873}{4000}$$

(iii) $P(F)=\dfrac{13}{20}, \quad P(F')=\dfrac{7}{20}$

$$P(S\mid F')=\frac{P(S\cap F')}{P(F')}=\frac{\frac{1}{2}\times\frac{2}{10}}{\frac{7}{20}}=\frac{2}{7}$$

$$P(R\mid F')=\frac{P(R\cap F')}{P(F')}=\frac{\frac{1}{4}\times\frac{6}{10}}{\frac{7}{20}}=\frac{3}{7}$$

$$P(L\mid F')=\frac{P(L\cap F')}{P(F')}=\frac{\frac{1}{4}\times\frac{4}{10}}{\frac{7}{20}}=\frac{2}{7}$$

Hence it is most likely that he has been to the river.

(iv) On a given sunday,

$P(\text{Both go to sea})=\dfrac{1}{2}\times\dfrac{1}{3}=\dfrac{1}{6}$

$P(\text{Both go to river})=\dfrac{1}{4}\times\dfrac{1}{3}=\dfrac{1}{12}$

$P(\text{Both go to lake})=\dfrac{1}{4}\times\dfrac{1}{3}=\dfrac{1}{12}$

$P(\text{Both meet})=\dfrac{1}{6}+\dfrac{1}{12}+\dfrac{1}{12}=\dfrac{1}{3}$

$P(\text{Both fail to meet on two sundays})=\dfrac{2}{3}\times\dfrac{2}{3}=\dfrac{4}{9}$

Hence the probability that they meet at least once is $=1-\dfrac{4}{9}=\dfrac{5}{9}$

14. (*b*)

	f	x	$d = \dfrac{x-450}{100}$	fd	fd^2
800 - 900	14	850	4	56	224
700 - 800	30	750	3	90	270
600 - 700	52	650	2	104	208
500 - 600	79	550	1	79	79
400 - 500	206	450	0	0	0
300 - 400	146	350	-1	-146	146
200 - 300	88	250	-2	-176	352
100 - 200	45	150	-3	-135	405
		660		-128	1684

(i) Mean $\quad \overline{x} = 450 + 100\left(\dfrac{-128}{660}\right)$

$\overline{x} = 469.39$

(ii) stadard deviation $\quad S = 100\sqrt{\dfrac{1684}{660} - \left(\dfrac{-128}{660}\right)^2}$

$S = 158.55$

(iii) Median class $= (400 - 500)$

Median $= 400 + \dfrac{100}{206}\left(\dfrac{660}{2} - 279\right)$

$= 400 + 100 \times \dfrac{51}{206}$

$= 424.75$

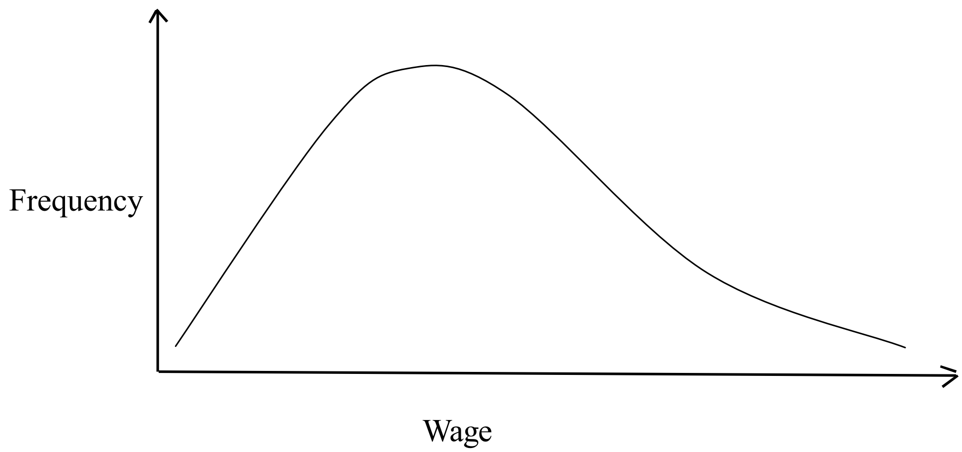
(iv) Coefficent of skewness $= 3\dfrac{(\text{mean - median})}{\text{standard deviation}}$

$= 3\dfrac{(469.39 - 424.75)}{158.55}$

$= 0.8446$

(v) Positively skewed curve.

Shape of the curve



15. (*a*) Use the following notation.

A: Adult

C: Child

M: Male

F: Female

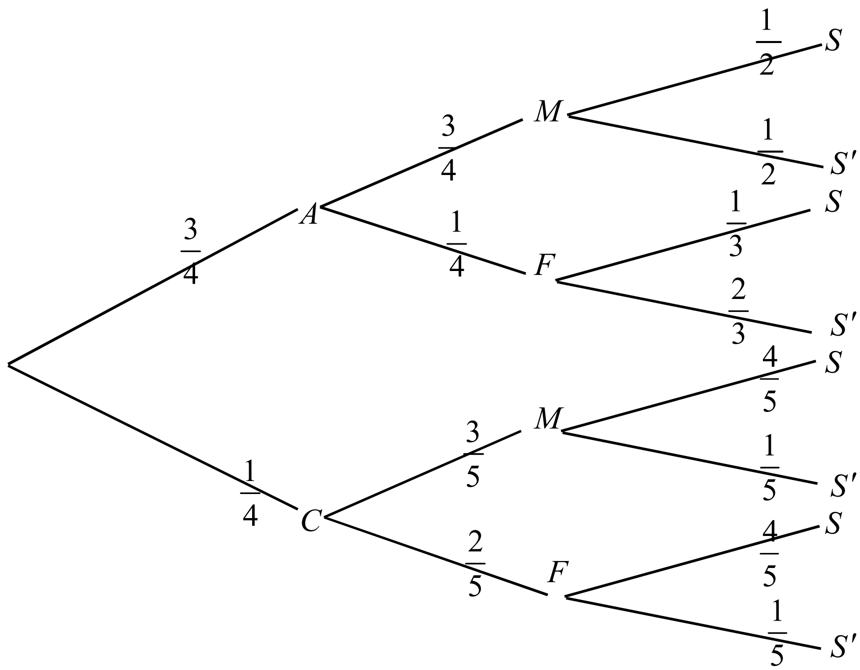
S: Using swimming pool.

$P(A)=\frac{3}{4}$, $P(C)=\frac{1}{4}$,

$P(M\mid C)=\frac{3}{5}$, $P(F\mid A)=\frac{1}{4}$, $P(F\mid C)=\frac{2}{5}$

$P(S\mid A\cap M)=\frac{1}{2}$, $P(S\mid A\cap F)=\frac{1}{3}$,

$P(S\mid C\cap M)=\frac{4}{5}$, $P(S\mid C\cap F)=\frac{4}{5}$



(i) $$P(S) = \left(\frac{3}{4}\times\frac{3}{4}\times\frac{1}{2}\right)+\left(\frac{3}{4}\times\frac{1}{4}\times\frac{1}{3}\right)+\left(\frac{1}{4}\times\frac{3}{5}\times\frac{4}{5}\right)+\left(\frac{1}{4}\times\frac{2}{5}\times\frac{4}{5}\right)$$

$$P(S) = \frac{9}{32}+\frac{1}{16}+\frac{3}{25}+\frac{2}{25}=\frac{87}{160}$$

(ii) $$P(F\mid S)=\frac{P(S\cap F)}{P(S)}$$

$$=\frac{\frac{3}{4}\times\frac{1}{4}\times\frac{1}{3}+\frac{1}{4}\times\frac{2}{5}\times\frac{4}{5}}{\frac{87}{160}}$$

$$=\frac{\frac{1}{16}+\frac{2}{25}}{\frac{87}{160}}$$

$$P(F\mid S)=\frac{114}{435}=0.262$$

(iii) $$P(C|M\cap S)=\frac{P(M\cap S\cap C)}{P(M\cap S)}$$

$$=\frac{\frac{1}{4}\times\frac{3}{5}\times\frac{4}{5}}{\frac{3}{4}\times\frac{3}{4}\times\frac{1}{2}+\frac{1}{4}\times\frac{3}{5}\times\frac{4}{5}}$$

$$=\frac{\frac{3}{25}}{\frac{9}{32}+\frac{3}{25}}$$

$$P(C\,|\,M\cap S)=\frac{3}{25}\times\frac{25\times 32}{321}=0.2999$$

(iv) $$P(A\cup F\,|\,S')=\frac{P\left[(A\cup F)\cap S'\right]}{P(S')}$$

$$P(S')=1-\frac{87}{160}=\frac{73}{160}$$

Now $(A\cup F)\cap S'=(A\cap S')\cup(F\cap S')$

Therefore $(A\cap S')\cup(F\cap S')=(A\cap M\cap S')\cup(A\cap F\cap S')\cup(C\cap F\cap S')$

Since all three events on R.H.S are mutnally exclusive.

Hence, $P\left[(A\cup F)\cap S'\right]=P\left[A\cap M\cap S'\right]+\left[A\cap F\cap S'\right]+P\left[C\cap F\cap S'\right]$

$$=\left(\frac{3}{4}\times\frac{3}{4}\times\frac{1}{2}\right)+\left(\frac{3}{4}\times\frac{1}{4}\times\frac{2}{3}\right)+\left(\frac{1}{4}\times\frac{2}{5}\times\frac{1}{5}\right)$$

$$P\left[(A\cup F)\cap S'\right]=\frac{9}{32}+\frac{1}{8}+\frac{1}{50}=\frac{341}{800}$$

$$P\left[(A\cup F)\,|\,S'\right]=\frac{P\left[(A\cup F)\cap S'\right]}{P(S')}$$

$$=\frac{\frac{341}{800}}{\frac{73}{160}}$$

$$=\frac{341}{365}$$

$$P\left[(A\cup F)\,|\,S'\right]=0.934$$

15. (*b*) $$\mu_1 = \frac{\sum_{1}^{n_1} x_i}{n_1}$$

$$\sum_{1}^{n_1} x_i = n_1\mu_1, \quad \sum_{1}^{n_2} x_i = n_2\mu_2$$

$$\sigma_1^{\,2} = \frac{\sum_{1}^{n_1} x_i^{\,2}}{n_1} - \mu_2^{\,1}$$

$$\sum_{1}^{n_1} x_i^{\,2} = n_1\sigma_1^{\,2} + n_1\mu_1^{\,2}$$

$$\sum_{1}^{n_2} x_i^2 = n_2\sigma_2^{\,2} + n_2\mu_2^{\,2}$$

Mean of the population $\overline{X} = \dfrac{n_1\mu_1 + n_2\mu_2}{n_1 + n_2}$

$$= \frac{n_1}{n_1 + n_2}\mu_1 + \frac{n_2}{n_1 + n_2}\mu_2$$

$$\overline{X} = \omega_1\mu_1 + \omega_2\mu_2$$

Where $\omega_1 = \dfrac{n_1}{n_1 + n_2}, \quad \omega_2 = \dfrac{n_2}{n_1 + n_2}$

Variancy the population $S^2 = \dfrac{\sum_{i=1}^{n_1+n_2} x_i}{n_1 + n_2} - \overline{X}^2$

$$S^2 = \frac{1}{n_1 + n_2}\left[\sum_{1}^{n_1} x_i^{\,2} + \sum_{1}^{n_2} x_i^{\,2}\right] - \overline{X}^2$$

$$S^2 = \frac{1}{n_1 + n_2}\left[n_1\sigma_1^{\,2} + n_2\sigma_2^{\,2} + n_1\mu_1^{\,2} + n_2\mu_2^{\,2}\right] - \left[\frac{n_1\mu_1 + n_2\mu_2}{n_1 + n_2}\right]^2$$

$$= \frac{n_1\sigma_1^{\,2}}{n_1 + n_2} + \frac{n_2\sigma_2^{\,2}}{n_1 + n_2} + \frac{(n_1 + n_2)}{(n_1 + n_2)^2}\left[n_1\mu_1^{\,2} + n_2\mu_2^{\,2}\right] - \left[\frac{n_1\mu_1 + n_2\mu_2}{n_1 + n_2}\right]^2$$

$$= \frac{n_1\sigma_1^{\ 2}}{n_1+n_2} + \frac{n_2\sigma_2^{\ 2}}{n_1+n_2} + \frac{1}{\left(n_1+n_2\right)^2}\begin{bmatrix} n_1\left(n_1+n_2\right)\mu_1^{\ 2} + n_2\left(n_1+n_2\right)\mu_2^{\ 2} \\ -2n_1n_2\mu_1\mu_2 - n_1^{\ 2}\mu_1^{\ 2} - n_2^{\ 2}\mu_2 \end{bmatrix}$$

$$= \omega_1\sigma_1^{\ 2} + \omega_2\sigma_2^{\ 2} + \frac{n_1n_2}{\left(n_1+n_2\right)^2}\left[\mu_1^{\ 2} + \mu^2 - 2\mu_1\mu_2\right]$$

$$S^2 = \omega_1\sigma_1^{\ 2} + \omega_2\sigma_2^{\ 2} + \omega_1\omega_2\left(\mu_1-\mu_2\right)^2$$

$$\overline{X} = \frac{\sum X}{n}$$

$$40 = \frac{\sum X}{20}$$

Wrong value of $\sum X = 800$

Correct value of $\sum X = 800 - 50 + 15$

$$= 765$$

Correct $\overline{X} = \frac{765}{20} = 38.25$

$$\sigma^2 = \frac{\sum x_i^2}{n} - \overline{X}^2$$

$$25 = \frac{\sum x_i^2}{20} - 40^2$$

Wrong $\sum x_i^2 = 500 + 1600 \times 20 = 32500$

Correct $\sum x_i^2 = 32500 - 2500 + 225$

$$= 30225$$

Correct $\sigma^2 = \frac{30225}{20} = 38.25^2$

$$= 1511.25 - 38.25^2$$

$$= 48.19$$

$$\sigma = \sqrt{48.19}$$

$$\sigma = 6.94$$

For the whole population $\mu = \dfrac{20\times 38.25 + 30\times 40.25}{20+30}$

$$= \frac{765 + 1207.5}{50}$$

$$\mu = \frac{1972.5}{50}$$

$$\mu = 39.45$$

$$\sigma^2 = \omega_1\sigma_1^{\ 2} + \omega_2\sigma_2^{\ 2} + \omega_1\omega_2\left(\mu_1 - \mu_2\right)^2$$

$$= \frac{20}{50}\times 6.94^2 + \frac{30}{50}\times 8^2 + \frac{20\times 30}{50\times 50}\left(40.25 - 30.25\right)^2$$

$$= \frac{2}{5}\times 48.19 + \frac{3}{5}\times 64 + \frac{6}{25}\times 4$$

$$= \frac{481.9 + 960 + 24}{25}$$

$$\sigma^2 = \frac{1465.9}{25}$$

$$= 58.636$$

$$\sigma = \sqrt{58.636}$$

$$\sigma = 7.65$$